Sparsity in Astrophysics

Astrostatistics meets Astroinformatics

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What is sparsity?

— representation of data in such a way that many data points are zero

What is sparsity?

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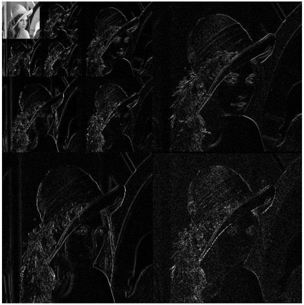
Sparsifying transform

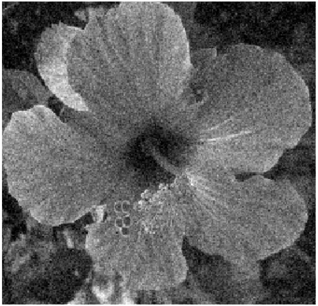


Why is sparsity useful?

— efficient characterisation of structure

Why is sparsity useful?

Add noise



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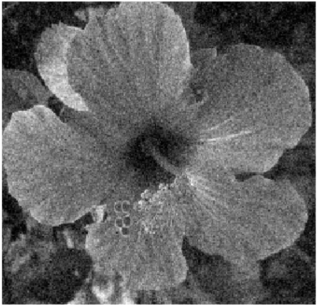

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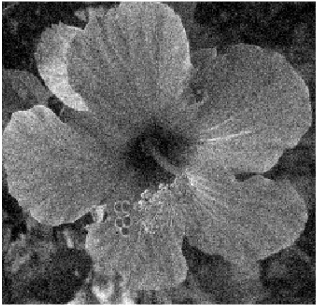

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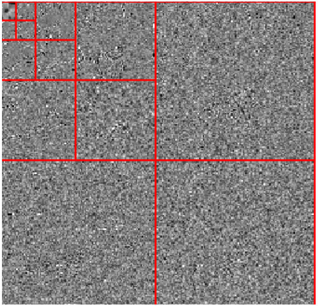

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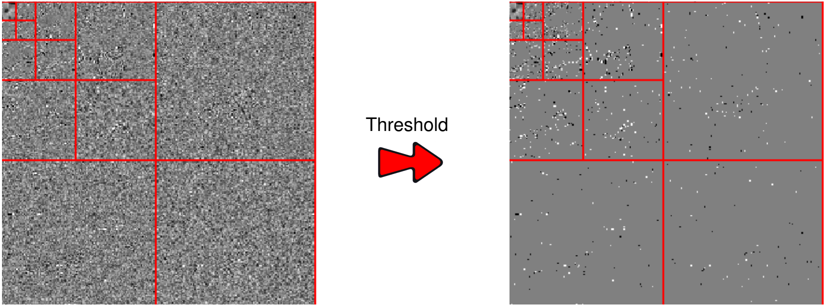

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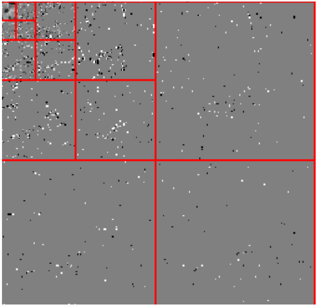

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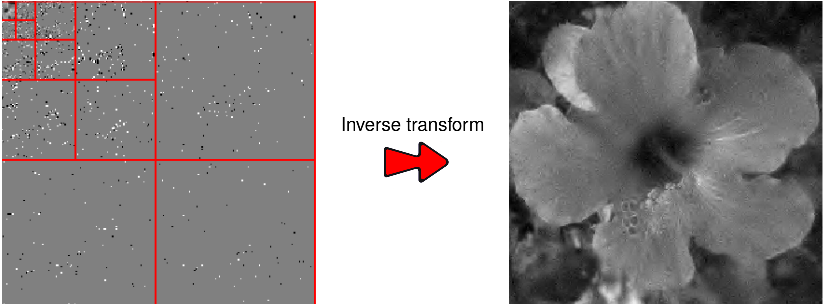

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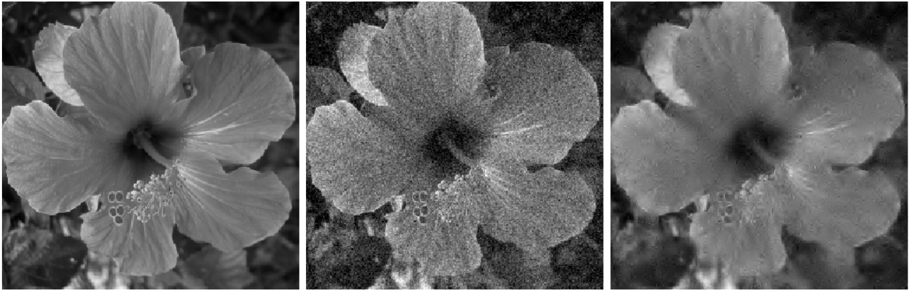

(a) Original (b) Noisy (SNR=14.1 dB) (c) Denoised (SNR=19.7 dB)

[Credit: http://www.ceremade.dauphine.fr/~peyre/numerical-tour/tours/denoisingwav_2_wavelet_2d/]

How can we construct sparsifying transforms?

— many signals in nature have spatially localised, scale-dependent features

How can we construct sparsifying transforms?

Fourier (1807)

Haar (1909)

Morlet and Grossman (1981)

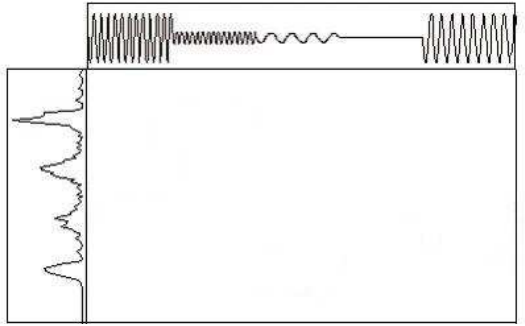

Figure: Fourier vs wavelet transform [Credit: `http://www.wavelet.org/tutorial/`]

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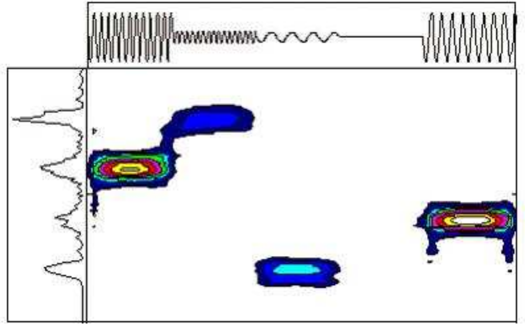

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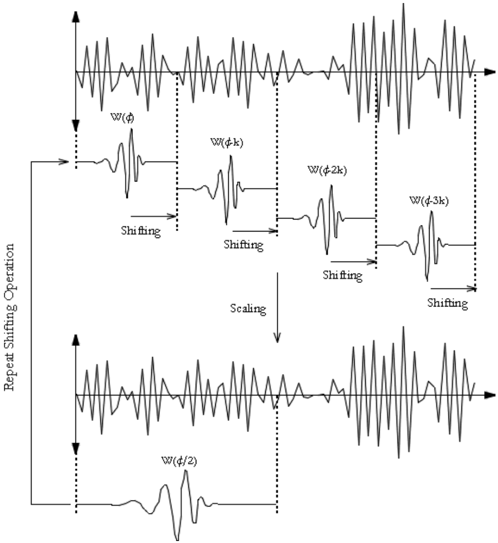

Figure: Wavelet scaling and shifting [Credit: `http://www.wavelet.org/tutorial/`]

Outline

1. Compressive Sensing
   - Introduction
   - Analysis vs Synthesis
   - Bayesian Interpretations

2. Radio Interferometric Imaging
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   - Sparsity Averaging (SARA)

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Compressive sensing

"Nothing short of revolutionary."

– National Science Foundation

- Developed by Emmanuel Candes and David Donoho (and others).
- Although many underlying ideas around for a long time.

(a) Emmanuel Candes

(b) David Donoho

Compressive sensing

- Next evolution of wavelet analysis $\rightarrow$ wavelets are a key ingredient.
- Mystery of JPEG compression (discrete cosine transform; wavelet transform).
- Move compression to the acquisition stage $\rightarrow$ compressive sensing.
- Acquisition versus imaging.

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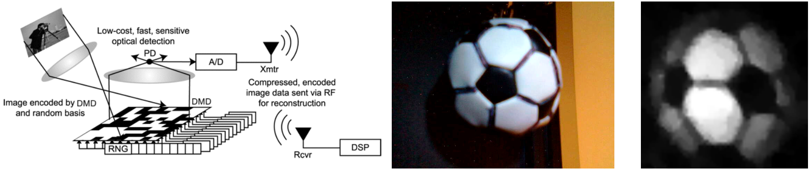

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Figure: Single pixel camera

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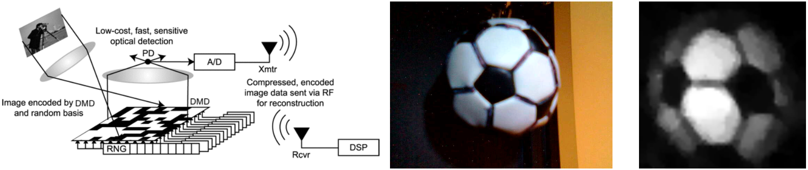

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An introduction to compressive sensing

Operator description

- Linear operator (linear algebra) representation of signal decomposition:

$$x(t) = \sum_i \alpha_i \Psi_i(t) \quad \rightarrow \quad \boldsymbol{x} = \sum_i \Psi_i \alpha_i = \begin{pmatrix} | \\ \Psi_0 \\ | \end{pmatrix} \alpha_0 + \begin{pmatrix} | \\ \Psi_1 \\ | \end{pmatrix} \alpha_1 + \cdots \quad \rightarrow \quad \boxed{\boldsymbol{x} = \Psi \boldsymbol{\alpha}}$$

- Linear operator (linear algebra) representation of measurement:

$$y_i = \langle x, \Phi_j \rangle \quad \rightarrow \quad \boldsymbol{y} = \begin{pmatrix} - \Phi_0 - \\ - \Phi_1 - \\ \vdots \end{pmatrix} \boldsymbol{x} \quad \rightarrow \quad \boxed{\boldsymbol{y} = \Phi \boldsymbol{x}}$$

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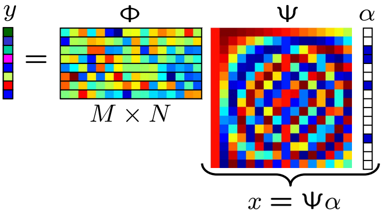

An introduction to compressive sensing

Promoting sparsity via ℓ_1 minimisation

- Ill-posed inverse problem:

$$\boldsymbol{y} = \Phi \boldsymbol{x} + \boldsymbol{n} = \Phi \Psi \boldsymbol{\alpha} + \boldsymbol{n} .$$

- Recall norms given by:

$$\|\alpha\|_0 = \text{no. non-zero elements} \qquad \|\alpha\|_1 = \sum_i |\alpha_i| \qquad \|\alpha\|_2 = \Big(\sum_i |\alpha_i|^2\Big)^{1/2}$$

- Solve by imposing a regularising prior that the signal to be recovered is sparse in Ψ, *i.e.* solve the following ℓ_0 optimisation problem:

$$\boldsymbol{\alpha}^\star = \arg\min_{\boldsymbol{\alpha}} \|\alpha\|_0 \;\text{ such that }\; \|\boldsymbol{y} - \Phi\Psi\boldsymbol{\alpha}\|_2 \leq \epsilon ,$$

where the signal is synthesising by $\boldsymbol{x}^\star = \Psi \boldsymbol{\alpha}^\star$.

- Solving this problem is difficult (combinatorial).

- Instead, solve the ℓ_1 optimisation problem (convex):

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An introduction to compressive sensing

Union of subspaces

- Space of sparse vectors given by the union of subspaces aligned with the coordinate axes.

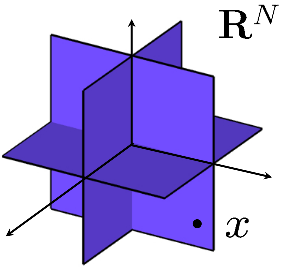

Figure: Space of the sparse vectors [Credit: Baraniuk]

An introduction to compressive sensing

RIP

- Solutions of ℓ_0 and ℓ_1 problems often the same.
- Restricted isometry property (RIP):

$$(1-\delta_{2K})\|\boldsymbol{x}_1-\boldsymbol{x}_2\|_2^2 \leq \|\Theta\boldsymbol{x}_1-\Theta\boldsymbol{x}_2\|_2^2 \leq (1+\delta_{2K})\|\boldsymbol{x}_1-\boldsymbol{x}_2\|_2^2 \,,$$

  for K-sparse $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, where $\Theta = \Phi\Psi$.
- Measurement must preserve geometry of sets of sparse vectors.

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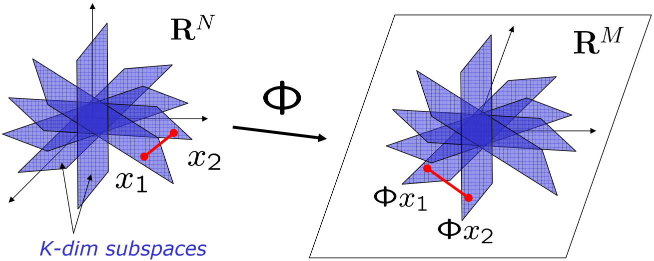

Figure: Measurement must preserve geometry of sets of sparse vectors. [Credit: Baraniuk]

An introduction to compressive sensing

Intuition

- Geometry of ℓ_0, ℓ_2 and ℓ_1 problems.

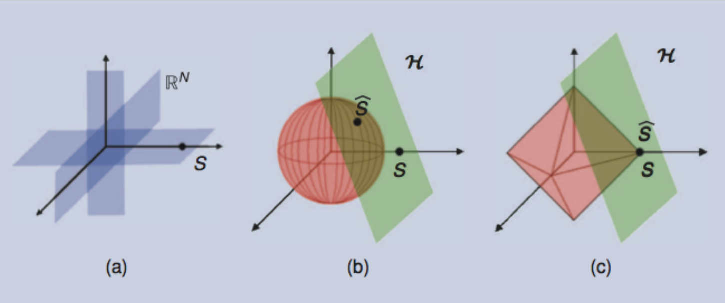

Figure: Geometry of (a) ℓ_0 (b) ℓ_2 and (c) ℓ_1 problems. [Credit: Baraniuk (2007)]

An introduction to compressive sensing

Coherence

- In the absence of noise, compressed sensing is exact!
- Number of measurements required to achieve exact reconstruction is given by

$$M \geq c\mu^2 K \log N ,$$

where K is the sparsity and N the dimensionality.

- The coherence between the measurement and sparsity basis is given by

$$\mu = \sqrt{N} \max_{i,j} |\langle \Psi_i, \Phi_j \rangle| .$$

- Robust to noise.

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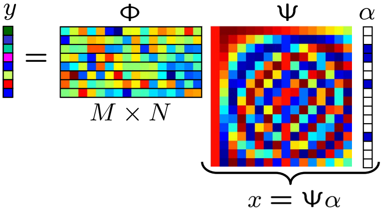

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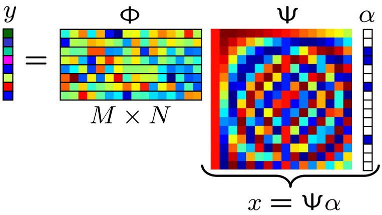

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Analysis vs synthesis

- Many new developments (*e.g.* analysis vs synthesis, structured sparsity).
- Typically sparsity assumption is justified by analysing example signals in terms of atoms of the dictionary.
- But this is different to synthesising signals from atoms.
- Suggests an analysis-based framework (Elad *et al.* 2007, Nam *et al.* 2012):

$$\boldsymbol{x}^\star = \underset{\boldsymbol{x}}{\arg\min} \, \|\Omega \boldsymbol{x}\|_1 \;\; \text{such that} \;\; \|\boldsymbol{y} - \Phi \boldsymbol{x}\|_2 \leq \epsilon \, .$$

analysis

- Contrast with synthesis-based approach:

$$\boldsymbol{x}^\star = \Psi \cdot \underset{\boldsymbol{\alpha}}{\arg\min} \, \|\boldsymbol{\alpha}\|_1 \;\; \text{such that} \;\; \|\boldsymbol{y} - \Phi \Psi \boldsymbol{\alpha}\|_2 \leq \epsilon \, .$$

synthesis

- For orthogonal bases $\Omega = \Psi^\dagger$ and the two approaches are identical.

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Redundant dictionaries

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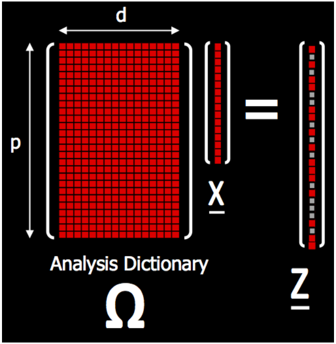

- Very different geometry to synthesis-based approach.

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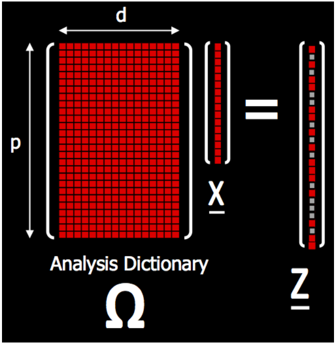


- Very different geometry to synthesis-based approach.

Analysis vs synthesis

Comparison

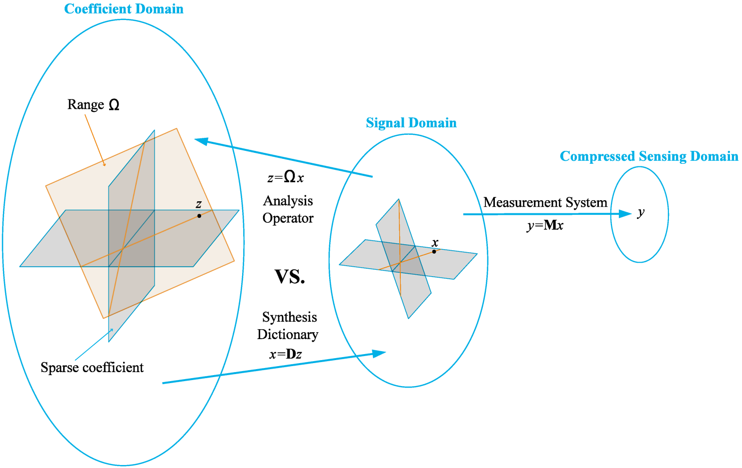
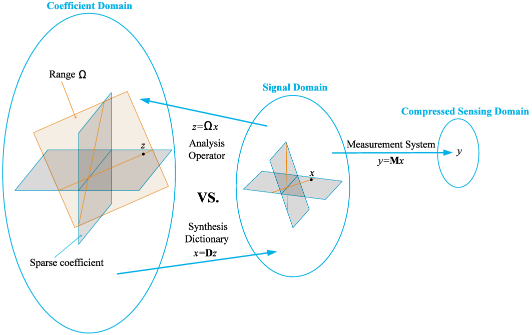


Figure: Analysis- and synthesis-based approaches [Credit: Nam *et al.* (2012)].

Analysis vs synthesis

Comparison

- Synthesis-based approach is more general, while analysis-based approach more restrictive.
- The more restrictive analysis-based approach may make it more robust to noise.
- The greater descriptive power of the synthesis-based approach may provide better signal representations (too descriptive?).



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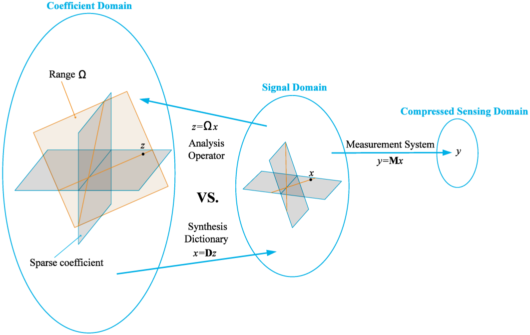

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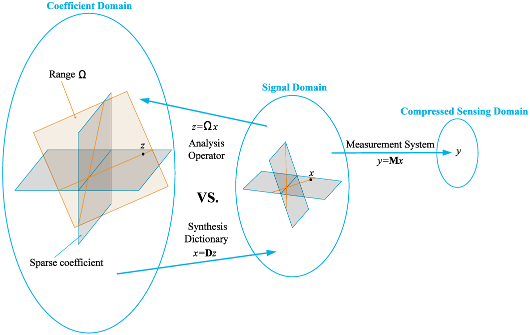

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Bayesian interpretations

One Bayesian interpretation of the synthesis-based approach

- Consider the inverse problem:

$$\boldsymbol{y} = \Phi \Psi \boldsymbol{\alpha} + \boldsymbol{n} \, .$$

- Assume Gaussian noise, yielding the likelihood:

$$\mathrm{P}(\boldsymbol{y} \,|\, \boldsymbol{\alpha}) \propto \exp\Bigl(\| \boldsymbol{y} - \Phi \Psi \boldsymbol{\alpha} \|_2^2 / (2\sigma^2) \Bigr) \, .$$

- Consider the Laplacian prior:

$$\mathrm{P}(\boldsymbol{\alpha}) \propto \exp\Bigl(-\beta \| \boldsymbol{\alpha} \|_1 \Bigr) \, .$$

- The maximum *a-posteriori* (MAP) estimate (with $\lambda = 2\beta\sigma^2$) is

$$\boldsymbol{x}^\star_{\text{MAP-Synthesis}} = \Psi \cdot \underset{\boldsymbol{\alpha}}{\arg\max}\, \mathrm{P}(\boldsymbol{\alpha} \,|\, \boldsymbol{y}) = \Psi \cdot \underset{\boldsymbol{\alpha}}{\arg\min}\, \| \boldsymbol{y} - \Phi \Psi \boldsymbol{\alpha} \|_2^2 + \lambda \| \boldsymbol{\alpha} \|_1 \, .$$

synthesis

- Signal may be ℓ_0-sparse, then solving ℓ_1 problem finds the correct ℓ_0-sparse solution!
- One possible Bayesian interpretation!

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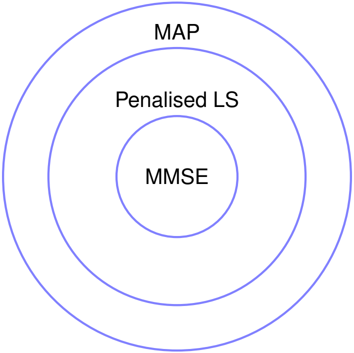
Bayesian interpretations

Other Bayesian interpretations of the synthesis-based approach

- Other Bayesian interpretations are also possible (Gribonval 2011).
- Minimum mean square error (MMSE) estimators
  - $\subset$ synthesis-based estimators with appropriate penalty function, *i.e.* penalised least-squares (LS)
  - $\subset$ MAP estimators



Bayesian interpretations

One Bayesian interpretation of the analysis-based approach

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analysis

- Identical to the synthesis-based approach if $\Omega = \Psi^\dagger$.

- But for redundant dictionaries, the analysis-based MAP estimate is

$$\boldsymbol{x}^\star_{\text{MAP-Analysis}} = \Omega^\dagger \cdot \arg\min_{\boldsymbol{\gamma} \in \text{column space } \Omega} \|\boldsymbol{y} - \Phi\Omega^\dagger\boldsymbol{\gamma}\|_2^2 + \lambda\|\boldsymbol{\gamma}\|_1 \ .$$

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Outline

1. Compressive Sensing
   - Introduction
   - Analysis vs Synthesis
   - Bayesian Interpretations

2. Radio Interferometric Imaging
   - Interferometric Imaging
   - Sparsity Averaging (SARA)

Next-generation of radio interferometry rapidly approaching

- Square Kilometre Array (SKA) first observations planned for 2019.
- Many other pathfinder telescopes under construction, *e.g.* LOFAR, ASKAP, MeerKAT, MWA.
- Broad range of science goals.
- New modelling and imaging techniques required to ensure the next-generation of interferometric telescopes reach their full potential.

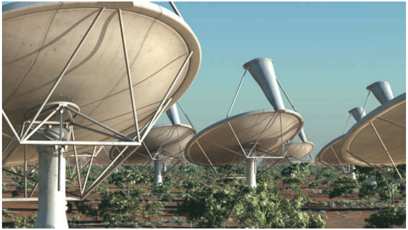

Figure: Artist impression of SKA dishes. [Credit: SKA Organisation]

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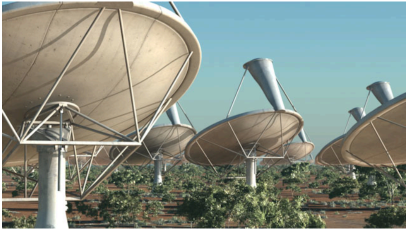

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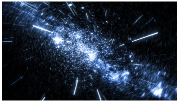
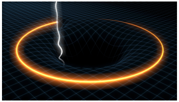
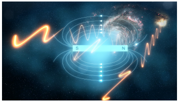


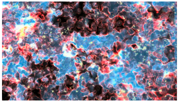
(a) Dark-energy



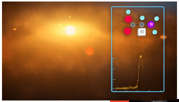
(b) GR



(c) Cosmic magnetism



(d) Epoch of reionization



(e) Exoplanets

Figure: SKA science goals. [Credit: SKA Organisation]

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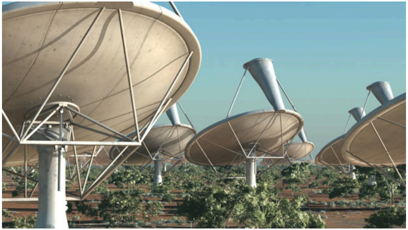

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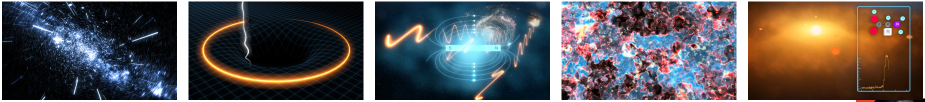

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Radio interferometric imaging

Inverse problem

- Consider the ill-posed inverse problem of radio interferometric imaging:

$$\boldsymbol{y} = \Phi \boldsymbol{x} + \boldsymbol{n} ,$$

  where $\boldsymbol{y}$ are the measured visibilities, Φ is the linear measurement operator, $\boldsymbol{x}$ is the underlying image and $\boldsymbol{n}$ is instrumental noise.

- Measurement operator $\Phi = \mathbf{M}\,\mathbf{F}\,\mathbf{C}\,\mathbf{A}$ may incorporate:
  - primary beam $\mathbf{A}$ of the telescope;
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Interferometric imaging: recover an image from noisy and incomplete Fourier measurements.

Radio interferometric imaging

Imaging

- Solve the interferometric imaging problem

$$\boldsymbol{y} = \Phi \boldsymbol{x} + \boldsymbol{n} \quad \text{with} \quad \Phi = \mathbf{M}\,\mathbf{F}\,\mathbf{C}\,\mathbf{A} \,,$$

by applying a prior on sparsity of the signal in a sparsifying dictionary Ψ.

- Basis pursuit (BP) denoising problem

$$\boldsymbol{\alpha}^\star = \arg\min_{\boldsymbol{\alpha}} \|\boldsymbol{\alpha}\|_1 \;\text{ such that }\; \|\boldsymbol{y} - \Phi\Psi\boldsymbol{\alpha}\|_2 \leq \epsilon \,,$$

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SARA for radio interferometric imaging

Algorithm

- Sparsity averaging reweighted analysis (SARA) for RI imaging (Carrillo, McEwen & Wiaux 2012)
- Consider a dictionary composed of a concatenation of orthonormal bases, i.e.

$$\Psi = \frac{1}{\sqrt{q}}[\Psi_1, \Psi_2, \dots, \Psi_q],$$

  thus $\Psi \in \mathbb{R}^{N \times D}$ with $D = qN$.
- We consider the following bases: Dirac (*i.e.* pixel basis); Haar wavelets (promotes gradient sparsity); Daubechies wavelet bases two to eight. $\Rightarrow$ concatenation of 9 bases.
- Promote average sparsity by solving the reweighted ℓ_1 analysis problem:

$$\min_{\bar{x} \in \mathbb{R}^N} \|W \Psi^T \bar{x}\|_1 \quad \text{subject to} \quad \|y - \Phi \bar{x}\|_2 \leq \epsilon \quad \text{and} \quad \bar{x} \geq 0 \,, \qquad \text{SARA}$$

  where $W \in \mathbb{R}^{D \times D}$ is a diagonal matrix with positive weights.
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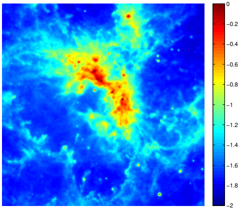
Results on simulations



(a) Original

SARA for radio interferometric imaging

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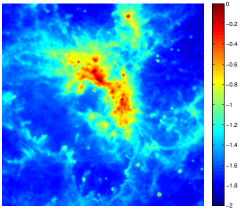

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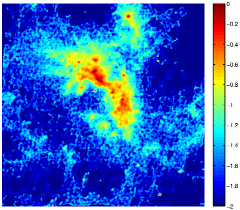
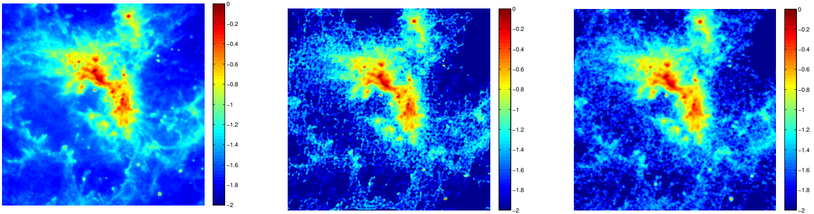


(b) BP (SNR=16.67 dB)

SARA for radio interferometric imaging

Results on simulations



(a) Original (b) BP (SNR=16.67 dB) (c) IUWT (SNR=17.87 dB)

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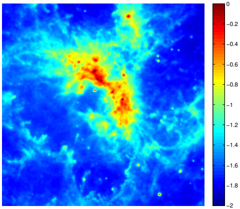

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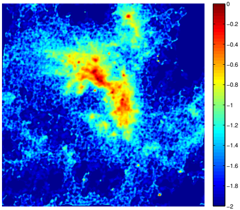

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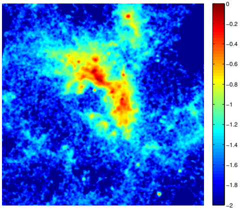

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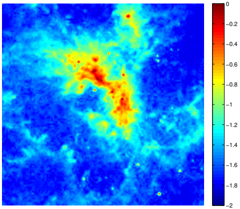
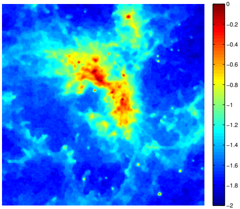
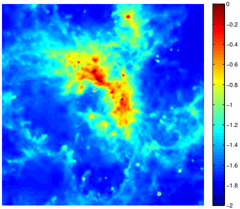


(d) BPDb8 (SNR=24.53 dB)



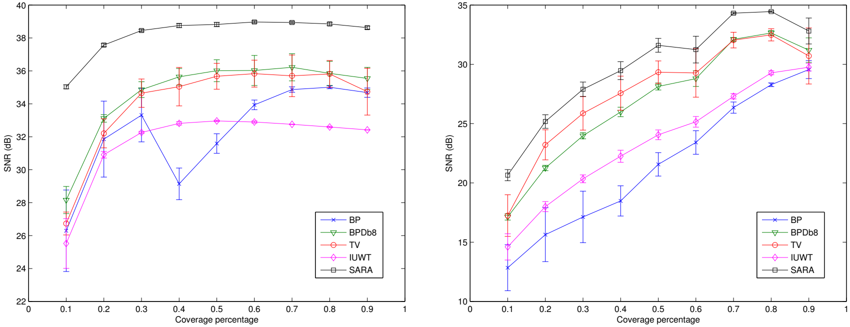
(e) TV (SNR=26.47 dB)



(f) SARA (SNR=29.08 dB)

SARA for radio interferometric imaging

Results on simulations



(a) M31 (b) 30Dor

Figure: Reconstruction fidelity vs visibility coverage.

SARA for radio interferometric imaging

Outlook

- Effectiveness of compressive sensing for radio interferometric imaging demonstrated.
- We have just released the PURIFY code to scale to the realistic setting.
- Includes state-of-the-art convex optimisation algorithms that support parallelisation.

Apply to observations made by real interferometric telescopes.

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PURIFY code http://basp-group.github.io/purify/

Next-generation radio interferometric imaging
Carrillo, McEwen, Wiaux

PURIFY is an open-source code that provides functionality to perform radio interferometric imaging, leveraging recent developments in the field of compressive sensing and convex optimisation.

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Many codes for application to cosmological data (CMB, LSS) available from:

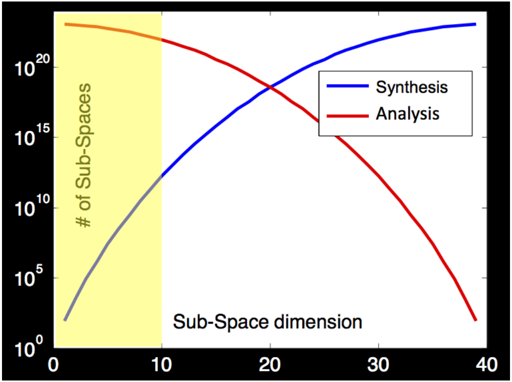
www.jasonmcewen.org

Extra Slides

Analysis vs synthesis

Size and number of subspaces

- For a given redundancy, size and number of subspaces very different between the analysis- and synthesis-approaches (Nam *et al.* 2012).



SARA for natural images

Results on simulations

(a) Original (b) Daubechies 8 (c) SARA

Figure: Lena reconstruction from 30% of Fourier measurements.

SARA for natural images

Results on simulations

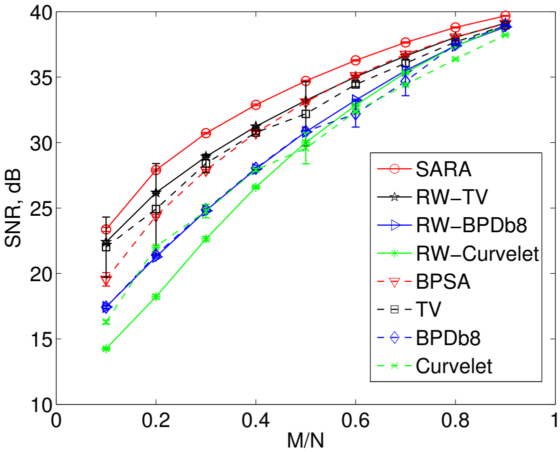


Figure: Reconstruction fidelity vs measurement ratio for Lena.

SARA for natural imaging

(a) Original (b) Daubechies 8 (c) SARA

Figure: Cameraman reconstruction from 30% of Fourier measurements.

SARA for natural imaging

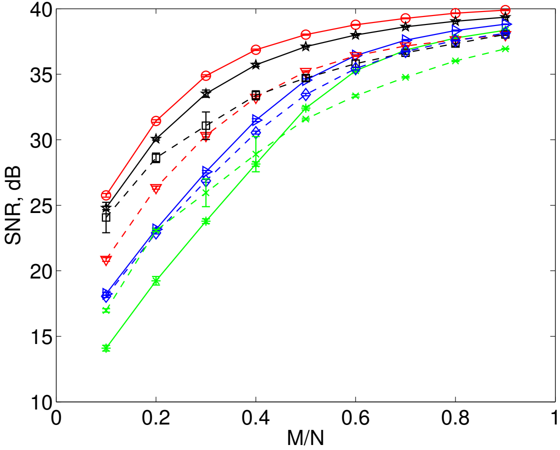
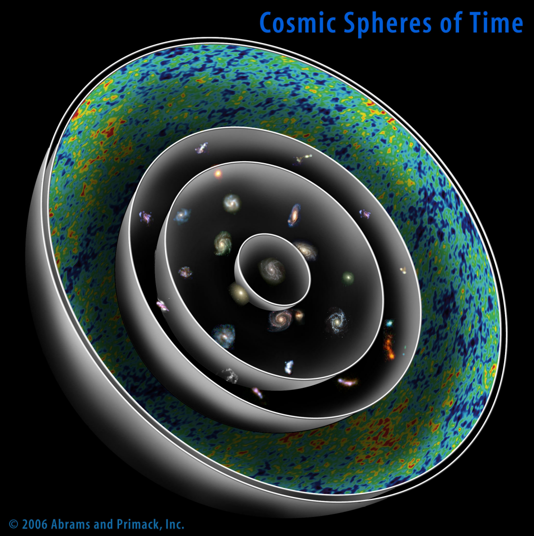
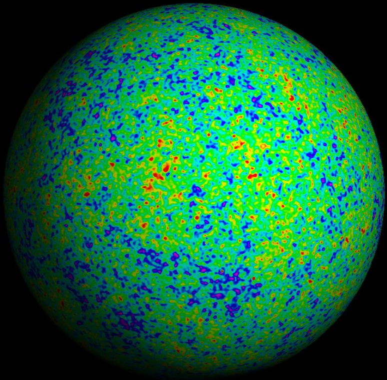


Figure: Reconstruction fidelity vs measurement ratio for Cameraman.

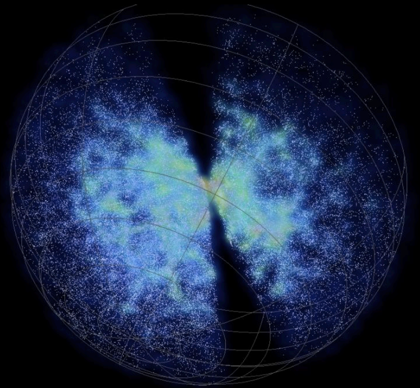
Observations made on the celestial sphere in cosmology



CMB observed on the sphere



LSS observed on the ball

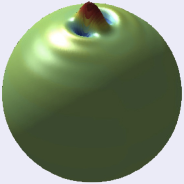


Scale-discretised wavelets on the sphere

Codes

S2DW code

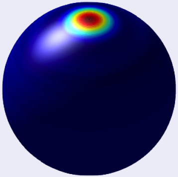
`http://www.s2dw.org`



Exact reconstruction with directional wavelets on the sphere
Wiaux, McEwen, Vandergheynst, Blanc (2008)

S2LET code

`http://www.s2let.org`

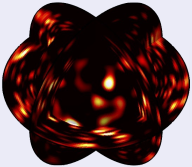


S2LET: A code to perform fast wavelet analysis on the sphere
Leistedt, McEwen, Vandergheynst, Wiaux (2012)

Fourier-LAGuerre wavelets (flaglets) on the ball

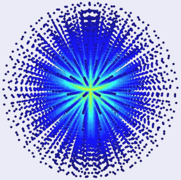
Codes

FLAGLET code http://www.flaglets.org



Exact wavelets on the ball
Leistedt & McEwen (2012)

FLAG code http://www.flaglets.org



FLAG: Fourier-Laguerre transform on the ball
Leistedt & McEwen (2012)

Radio interferometric inverse problem

- The complex visibility measured by an interferometer is given by

$$\boxed{y(\boldsymbol{u}, w) = \int_{D^2} A(\boldsymbol{l})\, x(\boldsymbol{l})\, C(\|\boldsymbol{l}\|_2)\, \mathrm{e}^{-\mathrm{i}2\pi \boldsymbol{u}\cdot\boldsymbol{l}}\, \frac{\mathrm{d}^2\boldsymbol{l}}{n(\boldsymbol{l})}},$$

visibilities

where the w-modulation $C(\|\boldsymbol{l}\|_2)$ is given by

$$\boxed{C(\|\boldsymbol{l}\|_2) \equiv \mathrm{e}^{\mathrm{i}2\pi w\left(1-\sqrt{1-\|\boldsymbol{l}\|^2}\right)}}.$$

w-modulation

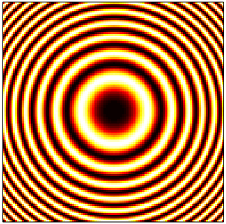
- Various assumptions are often made regarding the size of the field-of-view:
  - Small-field with $\|\boldsymbol{l}\|^2\, w \ll 1$ $\Rightarrow$ $C(\|\boldsymbol{l}\|_2) \simeq 1$
  - Small-field with $\|\boldsymbol{l}\|^4\, w \ll 1$ $\Rightarrow$ $C(\|\boldsymbol{l}\|_2) \simeq \mathrm{e}^{\mathrm{i}\pi w \|\boldsymbol{l}\|^2}$
  - Wide-field $\Rightarrow$ $C(\|\boldsymbol{l}\|_2) = \mathrm{e}^{\mathrm{i}2\pi w\left(1-\sqrt{1-\|\boldsymbol{l}\|^2}\right)}$

Spread spectrum effect
Review

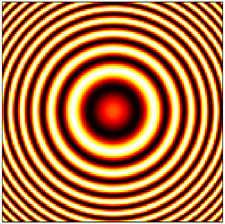
- Non-coplanar baselines and wide fields $\rightarrow$ w-modulation $\rightarrow$ spread spectrum effect (first considered by Wiaux *et al.* 2009b).
- The w-modulation operator $\mathbf{C}$ has elements defined by

$$C(l,m) \equiv \mathrm{e}^{\mathrm{i}2\pi w\left(1-\sqrt{1-l^2-m^2}\right)} \simeq \mathrm{e}^{\mathrm{i}\pi w\|\boldsymbol{l}\|^2} \quad \text{for} \quad \|\boldsymbol{l}\|^4\, w \ll 1 ,$$

giving rise to to a linear chirp.



(a) Real part



(b) Imaginary part

Figure: Chirp modulation.

Spread spectrum effect
Review

Spread spectrum effect in a nutshell

1. Radio interferometers take (essentially) Fourier measurements.
2. Recall, the coherence is the maximum inner product between measurement vectors and sparsifying atoms.
3. Thus, coherence is (essentially) the maximum of the Fourier coefficients of the atoms of the sparsifying dictionary.
4. w-modulation spreads the spectrum of the atoms of the sparsifying dictionary, reducing the maximum Fourier coefficient.
5. Spreading the spectrum reduces coherence, thus improving reconstruction fidelity.

- Consistent with findings of Carozzi et al. (2013) from information theoretic approach.
- Studied for constant w (for simplicity) by Wiaux *et al.* (2009b).
- Studied for varying w (with realistic images and various sparse representations) by Wolz *et al.* (2013).

Spread spectrum effect

Sparse w-projection

- Apply the w-projection algorithm (Cornwell *et al.* 2008) to shift the w-modulation through the Fourier transform:

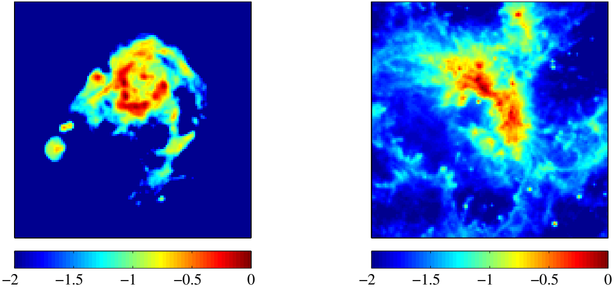
$$\Phi = \mathbf{M}\,\mathbf{F}\,\mathbf{C}\,\mathbf{A} \quad \Rightarrow \quad \Phi = \hat{\mathbf{C}}\,\mathbf{F}\,\mathbf{A} \, .$$

- Naively, expressing the application of the w-modulation in this manner is computationally less efficient that the original formulation but it has two important advantages.
- Different w for each (u, v), while still exploiting FFT.
- Many of the elements of $\hat{\mathbf{C}}$ will be close to zero.
- Support of w-modulation in Fourier space determined dynamically.

Spread spectrum effect for varying w

Results on simulations

- Perform simulations to assess the effectiveness of the spread spectrum effect in the presence of varying w.
- Consider idealised simulations with uniformly random visibility sampling.

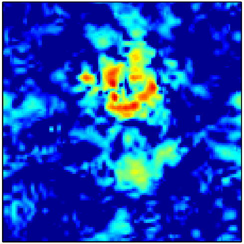


(a) M31 (b) 30Dor

Figure: Ground truth images in logarithmic scale.

Spread spectrum effect for varying w

Results on simulations

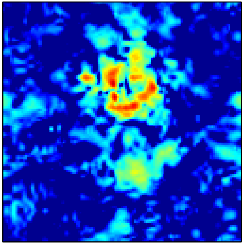


(a) $w_{\rm d} = 0 \rightarrow$ SNR$=5$dB

Figure: Reconstructed images of M31 for 10% coverage.

Spread spectrum effect for varying w

Results on simulations



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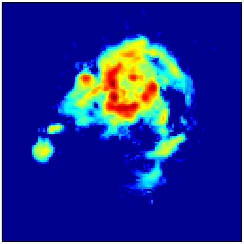

(c) $w_{\mathrm{d}} = 1 \rightarrow$ SNR$=$ 19dB

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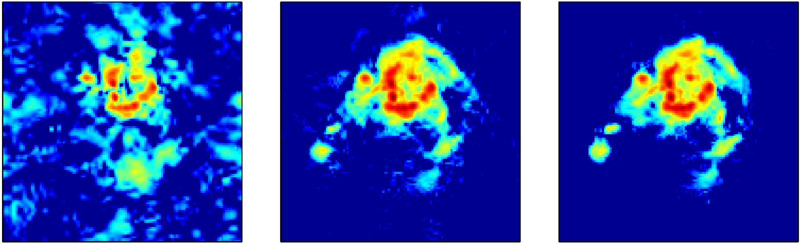
Spread spectrum effect for varying w

Results on simulations



(a) $w_\mathrm{d} = 0 \rightarrow$ SNR$=$ 5dB (b) $w_\mathrm{d} \sim \mathcal{U}(0, 1) \rightarrow$ SNR$=$ 16dB (c) $w_\mathrm{d} = 1 \rightarrow$ SNR$=$ 19dB

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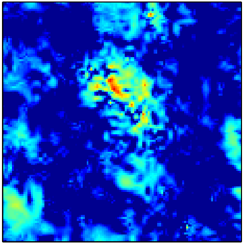
Spread spectrum effect for varying w

Results on simulations

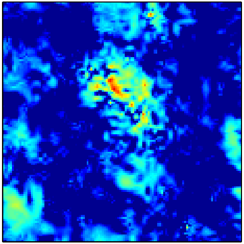


(a) $w_{\text{d}} = 0 \rightarrow$ SNR$=$ 2dB

Figure: Reconstructed images of 30Dor for 10% coverage.

Spread spectrum effect for varying w

Results on simulations



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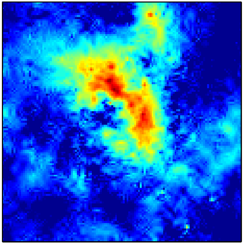

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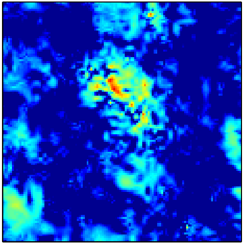

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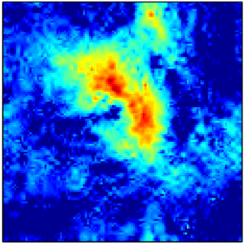
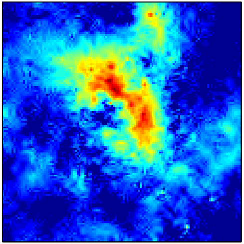


(b) $w_\text{d} \sim \mathcal{U}(0,1) \rightarrow$ SNR$=$ 12dB

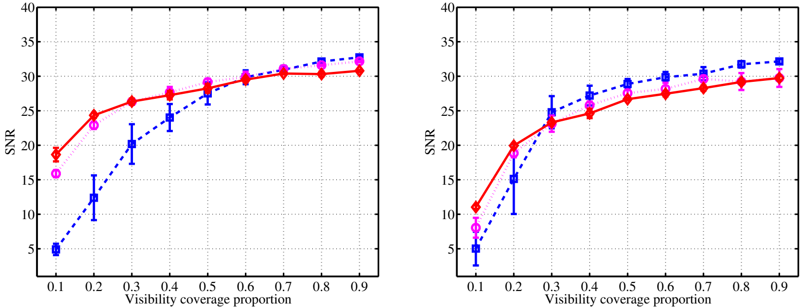


(c) $w_\text{d} = 1 \rightarrow$ SNR$=$ 15dB

Figure: Reconstructed images of 30Dor for 10% coverage.

Spread spectrum effect for varying w

Results on simulations



(a) Daubechies 8 (Db8) wavelets (b) Dirac basis

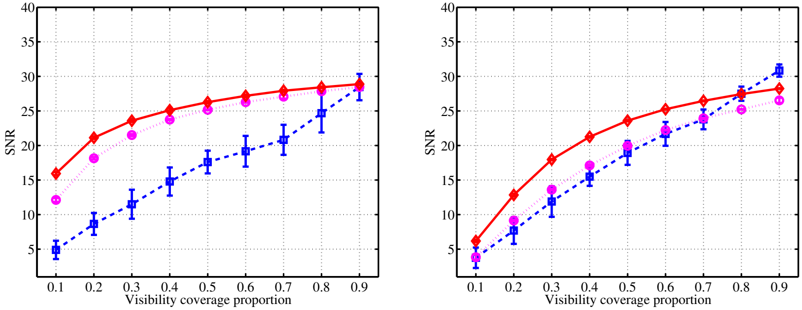
Figure: Reconstruction fidelity for M31.

Improvement in reconstruction fidelity due to the spread spectrum effect for varying w is almost as large as the case of constant maximum w!

- As expected, for the case where coherence is already optimal, there is little improvement.

Spread spectrum effect for varying w

Results on simulations



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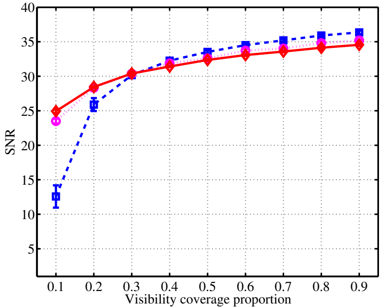
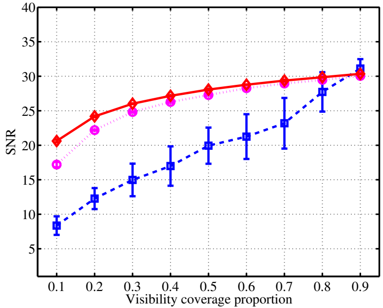


(a) M31



(b) 30 Dor

Figure: Reconstruction fidelity using SARA.

Improvement in reconstruction fidelity due to the spread spectrum effect for varying w is almost as large as the case of constant maximum w!

- As expected, for the case where coherence is already optimal, there is little improvement.

Supporting continuous visibilities

Algorithm

- Ideally we would like to model the continuous Fourier transform operator

$$\boxed{\Phi = \mathbf{F}^{\mathrm{c}}} \; .$$

- But this is impractically slow!
- Incorporated gridding into our CS interferometric imaging framework.
- Work of Rafael Carrillo, in collaboration with Wiaux and McEwen (see Carrillo, McEwen, Wiaux 2013).
- Model with measurement operator

$$\boxed{\Phi = \mathbf{G}\,\mathbf{F}\,\mathbf{D}\,\mathbf{Z}} \; ,$$

where we incorporate:

  - convolutional gridding operator $\mathbf{G}$;
  - fast Fourier transform $\mathbf{F}$;
  - normalisation operator $\mathbf{D}$ to undo the convolution gridding;
  - zero-padding operator $\mathbf{Z}$ to upsample the discrete visibility space.

Supporting continuous visibilities

Results on simulations

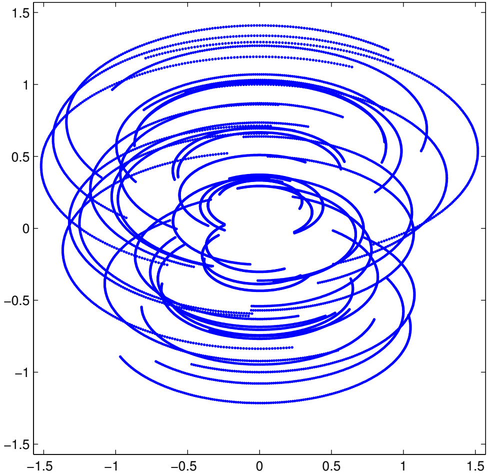


Figure: Coverage

Supporting continuous visibilities
Results on simulations

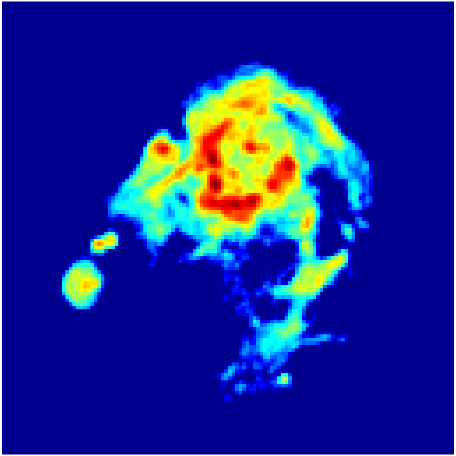


Figure: M31 (ground truth).

Supporting continuous visibilities
Results on simulations

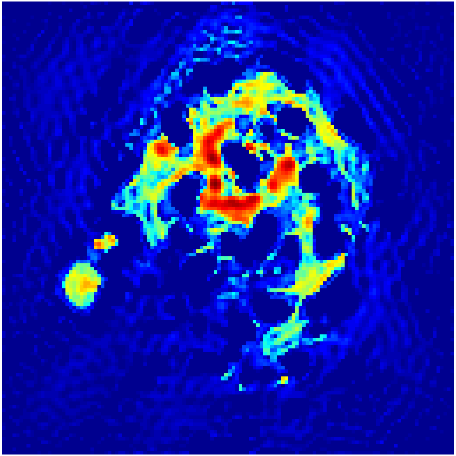


Figure: Dirac basis ("CLEAN") $\rightarrow$ SNR$=8.2$dB.

Supporting continuous visibilities

Results on simulations

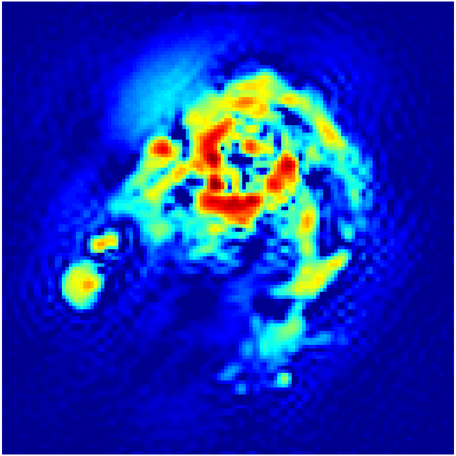


Figure: Db8 wavelets ("MS-CLEAN") → SNR= 11.1dB.

Supporting continuous visibilities
Results on simulations

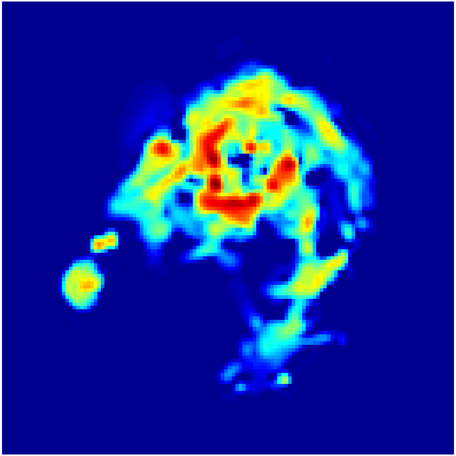


Figure: SARA $\rightarrow$ SNR$= 13.4$dB.