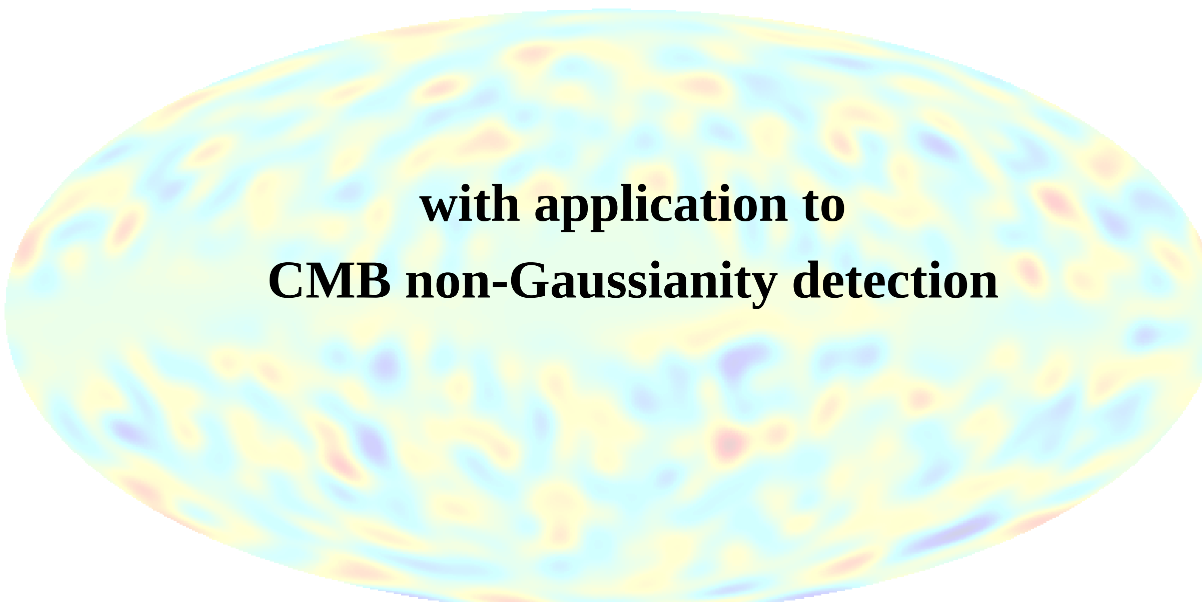
A Fast Directional Spherical Wavelet Transform

for the Analysis of Cosmological Data



with application to
CMB non-Gaussianity detection

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XXXIXth Recontres de Moriond
Exploring the Universe

Overview

- Directional Continuous Spherical Wavelet Transform (CSWT)
- Fast algorithm
- Deviation from Gaussianity in WMAP anisotropies using spherical wavelet analysis
  - Isotropic spherical Mexhat wavelets
  - Directional spherical real Morlet wavelets
- Conclusions and future work

- Follow the formalisation of Antoine and Vandergheynst (1999)
- Notation

$$S^2, \quad \omega = (\theta, \phi), \quad d\mu(\omega) = \sin(\theta) d\theta d\phi$$

- Rotations

$$(\mathcal{R}_\rho f)(\omega) = f(\rho^{-1}\omega) \quad \text{where } f \in L^2(S^2), \ \rho \in SO(3)$$

- Dilations

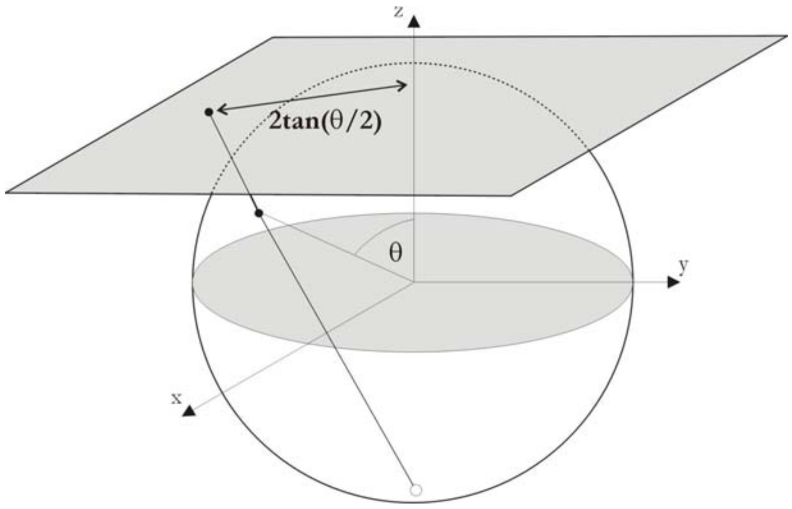
$$(\mathcal{D}_a f)(\omega) = f_a(\omega) = \sqrt{\lambda(a,\theta)}\, f(\omega_{1/a}), \quad a \in \mathbb{R}_*^+$$

where $\omega_a = (\theta_a, \phi)$,

$$\tan(\theta_a/2) = a \tan(\theta/2)$$

enforce

$$\|\mathcal{D}_a f\|_2 = \|f\|_2 \quad \Rightarrow \quad \lambda(a,\theta)$$

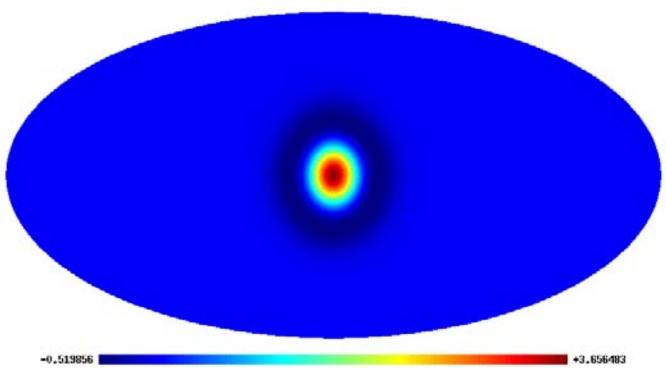


Inverse Stereographic Projection

- Project Euclidean planar wavelets onto the sphere

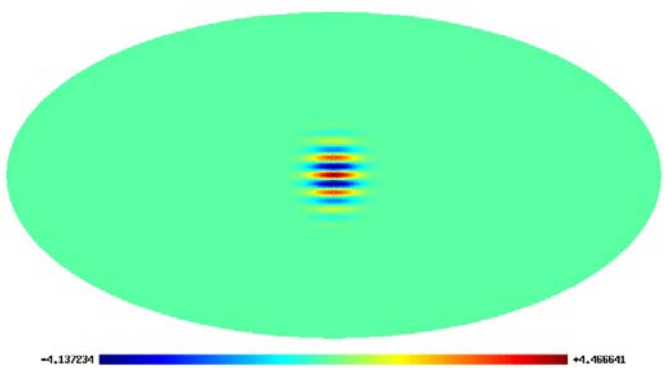
$$\psi_{S^2}(\theta, \phi) = \frac{2}{1 + \cos(\theta)} \psi_{\mathbb{R}^2}(2 \tan(\theta/2), \phi)$$

Spherical Mexhat wavelet



$$\psi_{\mathbb{R}^2}(\mathbf{x}) = \frac{1}{\sqrt{2\pi}}(1 - \parallel \mathbf{x} \parallel^2) e^{-\frac{\|\mathbf{x}\|^2}{2}}$$

Spherical real Morlet wavelet



$$\psi_{\mathbb{R}^2}(\mathbf{x}) = \cos(\mathbf{k} \cdot \mathbf{x}) e^{-\|\mathbf{x}\|^2}$$

- Continuous Spherical Wavelet Transform

$$S(a, \alpha, \beta, \gamma) = \int_{S^2} \overline{(\mathcal{R}_{\alpha,\beta,\gamma}\psi_a)(\omega')} s(\omega') d\mu(\omega')$$

Fast algorithm

- Based on fast spherical convolution of Wandelt and Gorski (2001)
- Harmonic formulation

$$S(\alpha,\beta,\gamma) = \sum_{l=0}^{l_{max}} \sum_{m=-l}^{l} \sum_{m'=-l}^{l} \left(\overline{D^l_{mm'}(\alpha,\beta,\gamma)\psi_{lm}} \right) s_{lm}$$

Wigner rotation matrix: $D^l_{mm'}(\alpha,\beta,\gamma) = e^{-im\alpha}\, d^l_{mm'}(\beta)\, e^{-im'\gamma}$

Factor rotation: $\mathcal{R}_{\alpha,\beta,\gamma} = \mathcal{R}_{\alpha-\pi/2,\,-\pi/2,\,\beta}\, \mathcal{R}_{0,\,\pi/2,\,\gamma+\pi/2}$

$$S(\alpha,\beta,\gamma) = \sum_{l=0}^{l_{max}} \sum_{m=-l}^{l} \sum_{m'=-l}^{l} \sum_{m''=-min(m_{max},l)}^{min(m_{max},l)} d^l_{m'm}(\pi/2)\, d^l_{m'm''}(\pi/2)\, \overline{\psi_{lm''}} s_{lm}\, e^{i[m(\alpha-\pi/2)+m'\beta+m''(\gamma+\pi/2)]}$$

Fast algorithm

- Discretise, interchange order of summation and shift indices

$$S[n_\alpha, n_\beta, n_\gamma] = e^{-i2\pi[n_\alpha l_{max}/N_\alpha + n_\beta l_{max}/N_\beta + n_\gamma m_{max}/N_\gamma]}$$

$$\sum_{j=0}^{N_\alpha - 1} \sum_{j'=0}^{N_\beta - 1} \sum_{j''=0}^{N_\gamma - 1} t_{j,j',j''} \, e^{i2\pi[j n_\alpha/N_\alpha + j' n_\beta/N_\beta + j'' n_\gamma/N_\gamma]}$$

where $t_{m+l_{max},\, m'+l_{max},\, m''+m_{max}} = e^{i(m''-m)\pi/2}$

$$\sum_{l=max(|m|,|m'|,|m''|)}^{l_{max}} d^l_{m'm}(\pi/2) \, d^l_{m'm''}(\pi/2) \, \overline{\psi_{lm''}} s_{lm}$$

- Evaluate $t_{j,j',j''}$ first, then use FFT to simultaneously evaluate all other summations

Algorithms

Comparison

Theoretical complexity

Algorithm	Complexity
Direct	$\mathcal{O}(L^4 N_\gamma)$
Semi-fast	$\mathcal{O}(L^3 \log_2(L) N_\gamma)$
Fast	$\mathcal{O}(L^3 N_\gamma)$

$\Rightarrow$ Saving: $\mathcal{O}(L) \sim \mathcal{O}(\sqrt{N})$

Typical execution times

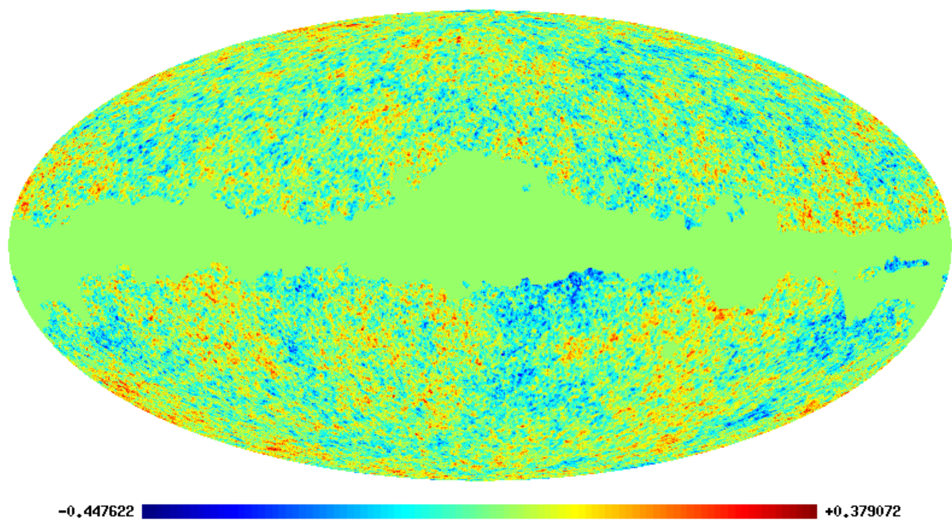
$\mathrm{N_{side}}$	Execution time (min:sec)		
	Direct	Semi-fast	Fast
8	00:01.19	00:01.12	00:00.01
16	00:18.60	00:17.38	00:00.04
32	05:01.48	04:43.06	00:00.21
256	-	-	01:54.15

Sun Fire 280R Server
Dual UltraSPARC III 900MHz Processors
4GB Memory

Non-Gaussianity

Data preprocessing

- Combine foreground corrected WMAP Q-V-W bands
  $\Rightarrow$ single signal-to-noise ratio enhanced map
- Down-sample to N_{side}=256 resolution
- Apply Kp0 mask to remove Galactic plane and point source contamination



Non-Gaussianity

- Create Gaussian CMB realisation from theoretical power spectrum (monopole and dipole removed)
- Convolve CMB realisation with each Q-V-W beam
- Combine simulated Q-V-W bands in same manner as WMAP data
- Analyse Gaussian simulated maps in same manner as WMAP data to construct Gaussian confidence regions

Non-Gaussianity

- Take CSWT of data at following scales:

Scales	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}
Arcmin	14	25	50	75	100	150	200	250	300	400	500

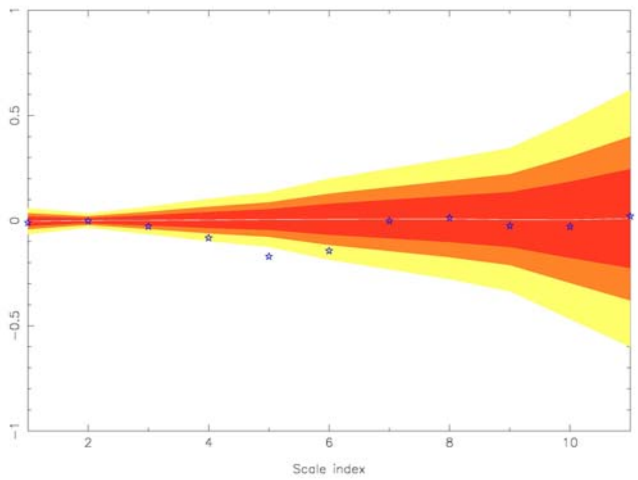
- Construct and apply extended coefficient mask
- Consider statistics of wavelet coefficients to detect deviations from Gaussianity
  - Skewness
  - Kurtosis
- Apply identical analysis to Gaussian simulated data to construct confidence regions

Non-Gaussianity

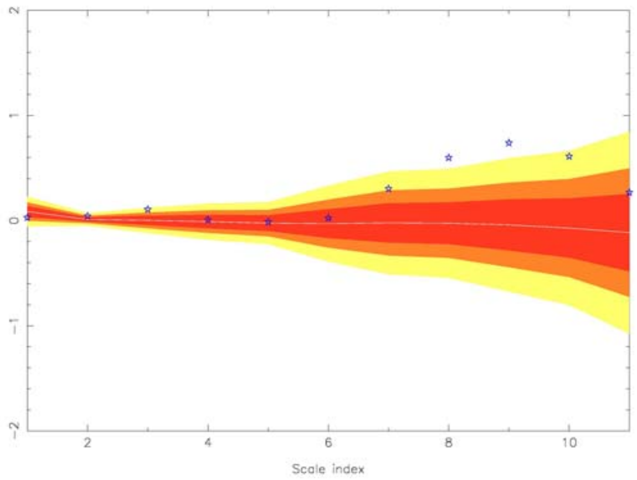
Mexhat wavelet results

- Reproduced results of Vielva et al. (2003)
- Deviation from Gaussianity detected in kurtosis at R_8 and R_9
- Possible non-Gaussianity in skewness at R_5 also detected

Skewness



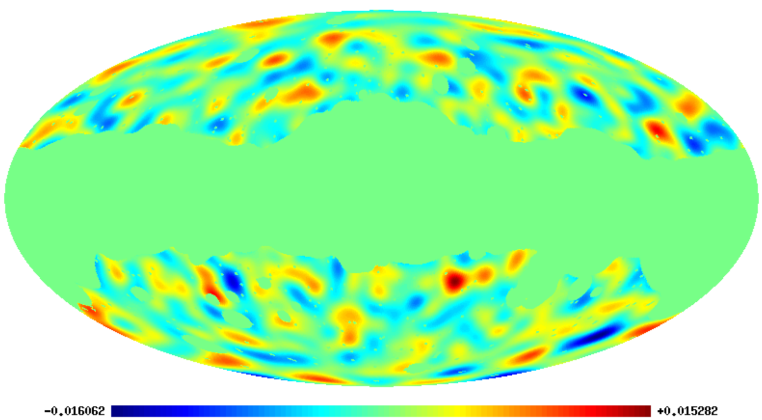
Kurtosis



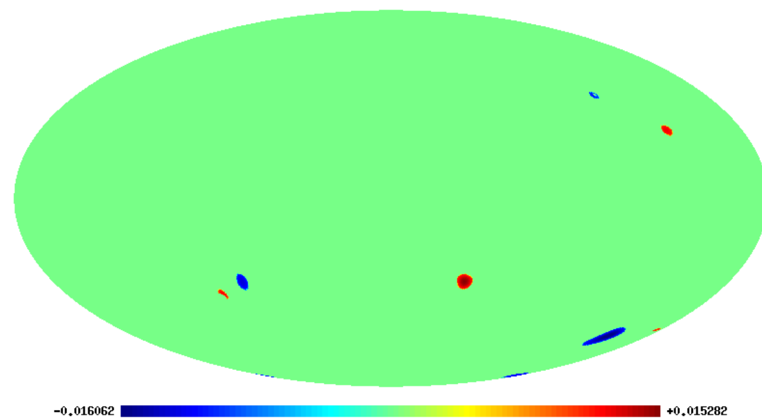
Non-Gaussianity

Mexhat wavelet results

- Spatial localisation of most likely deviations from Gaussianity
- Threshold R_8 coefficients below 3σ



Wavelet coefficients



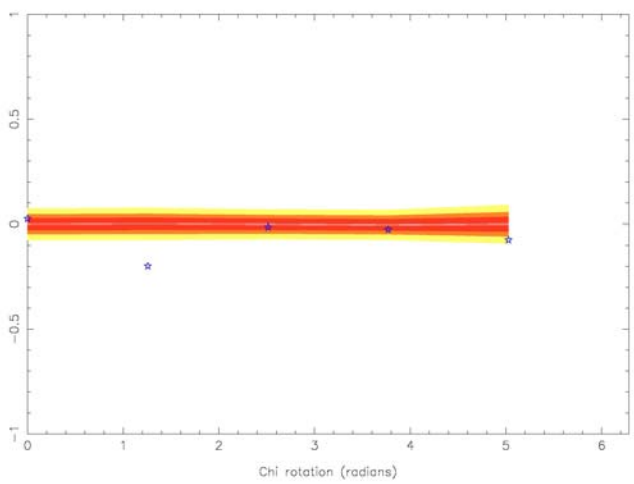
Thresholded wavelet coefficients

Non-Gaussianity

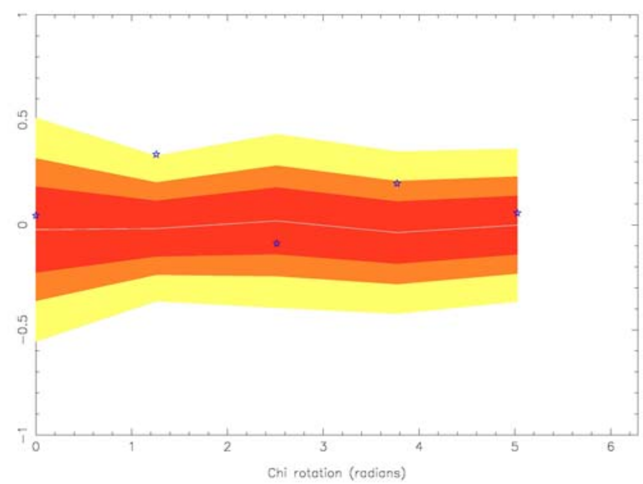
Morlet wavelet results

- Directional selectivity (consider 5 orientations)
- No significant deviations from non-Gaussianity at previous scales
- Deviation from Gaussianity detected at *R*=*750* arcmin, $\gamma=72^\circ$

Skewness



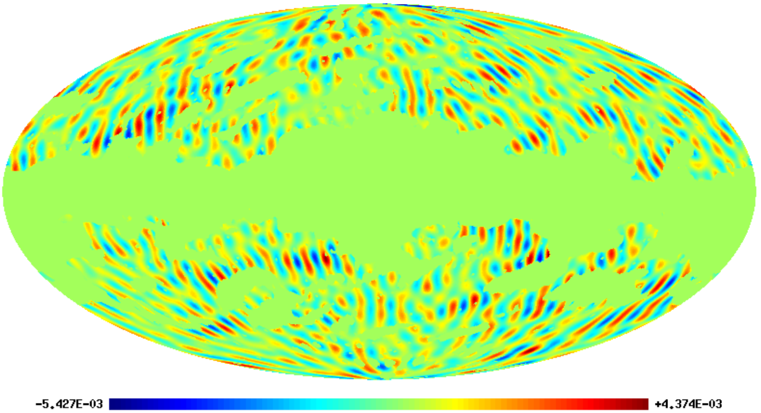
Kurtosis



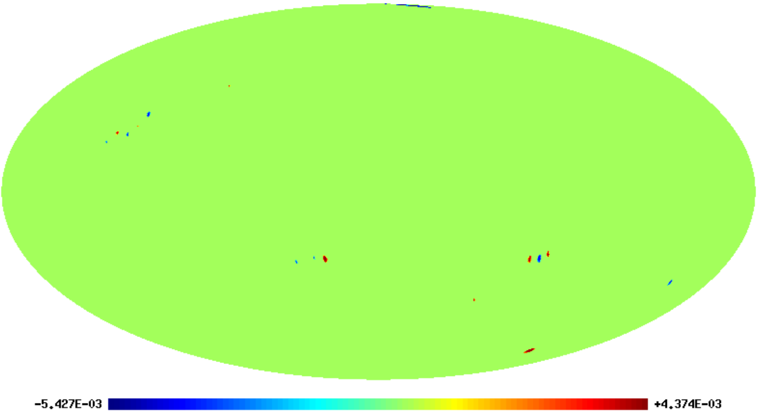
Non-Gaussianity

Morlet wavelet results

- Spatial and directional localisation of most likely deviations from Gaussianity
- Threshold $R=750$ arcmin, $\gamma=72°$ coefficients below 3σ



Wavelet coefficients



Thresholded wavelet coefficients

Summary

- Directional Continuous Spherical Wavelet Transform
- Fast algorithms: $\mathcal{O}(L) \sim \mathcal{O}(\sqrt{N})$ saving
- Possible deviation from Gaussianity detected in WMAP
- Future work
  - Further analysis of coefficient statistics and statistical tests
  - Consider other spherical wavelets, scales, directions
  - Other applications of CSWT (e.g. compact object/defect detection using matched wavelets)

- Acknowledgements
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