Cosmological Image Processing

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Mullard Space Science Laboratory (MSSL)
University College London (UCL)

Cosmology @ MSSL

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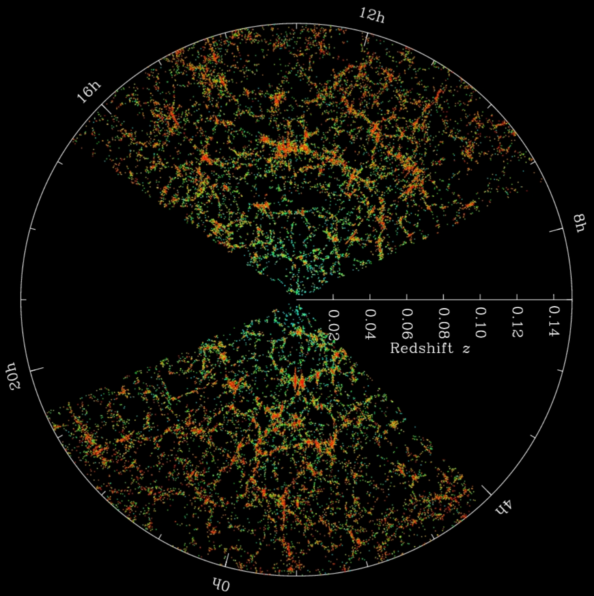
LSS fly through


12h
16h
8h
20h
4h
0h
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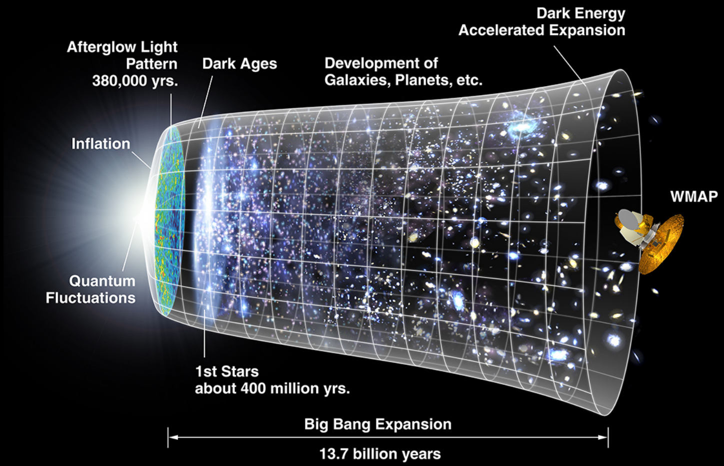
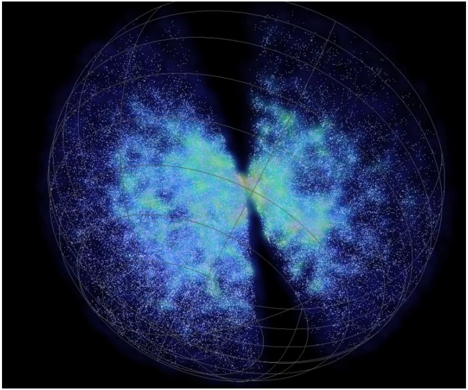
Redshift z

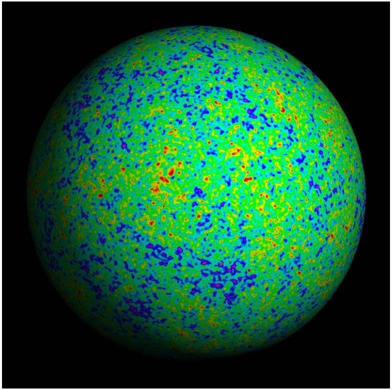
We have entered an era of concordance cosmology.



Cosmological observations



(a) Large-scale structure [Credit: SDSS]



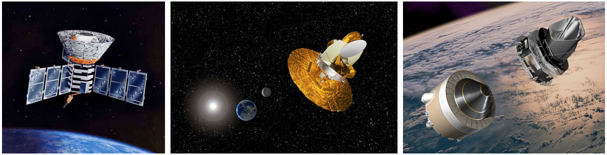
(b) Cosmic microwave background [Credit: WMAP]

Figure: Cosmological observations

Telescopes and satellites

(a) SDSS (b) DES (c) Euclid

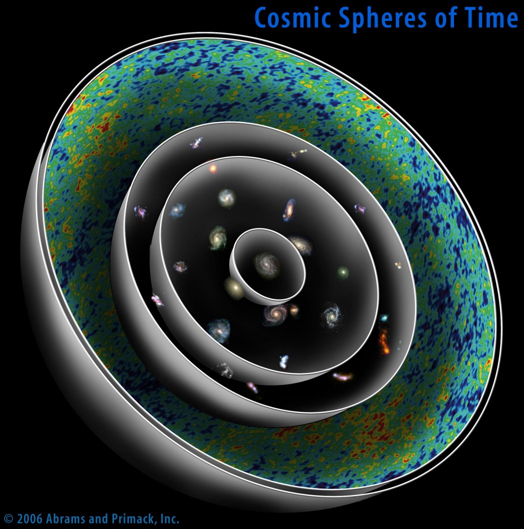
Figure: LSS observations



(a) COBE (b) WMAP (c) Planck

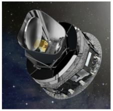
Figure: Full-sky CMB observations

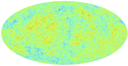
Observations made on the celestial sphere



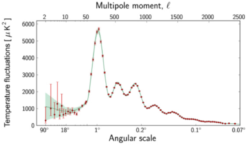
Precision cosmology

Case study: CMB

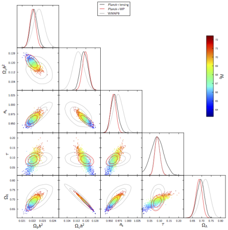




Observation



$$P(\Theta|d, M) \propto P(d|\Theta, M)P(\Theta|M)$$



Bayesian Inference

$$\Theta^\star = (A_s^\star, n_s^\star, \tau^\star, \Omega_\mathrm{b}^\star h^2, \Omega_\mathrm{c}^\star h^2, \Omega_\Lambda^\star)$$

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Theory

Precision cosmology

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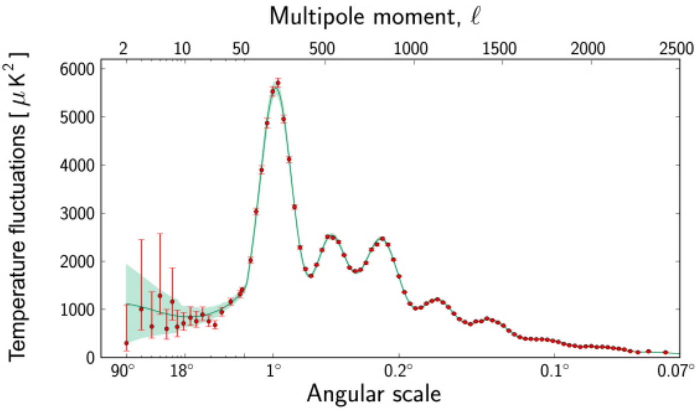
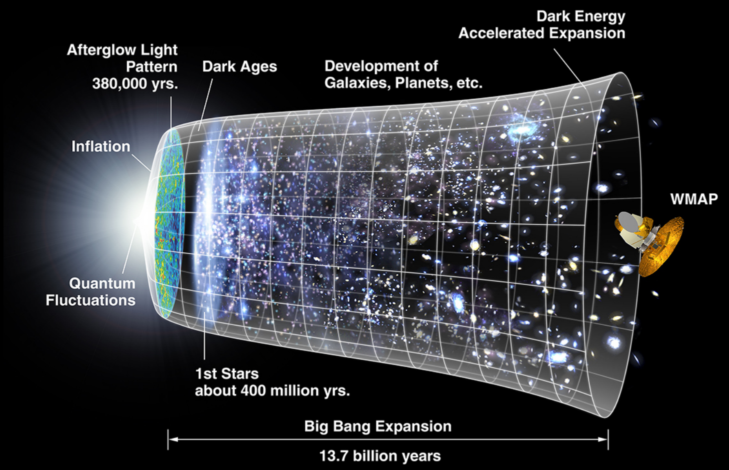


Outstanding questions



Outline

Dark energy

- Universe consists of ordinary baryonic matter, cold dark matter and dark energy.
- Dark energy represents energy density of empty space, which acts as a repulsive force.
- Strong evidence for dark energy exists but we know very little about its nature and origin.
- A consistent model in the framework of particle physics lacking.

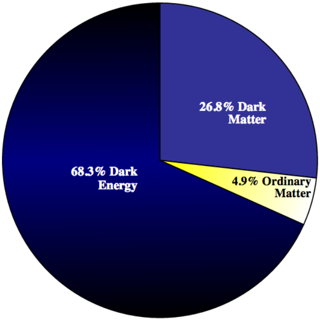


Figure: Content of the Universe [Credit: Planck]

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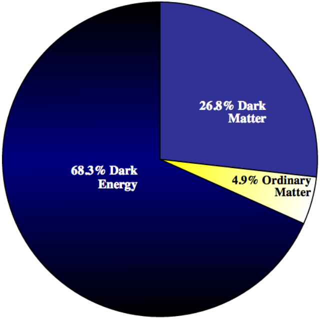

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Integrated Sachs Wolfe Effect

Analogy

(no dark energy)

(with dark energy)

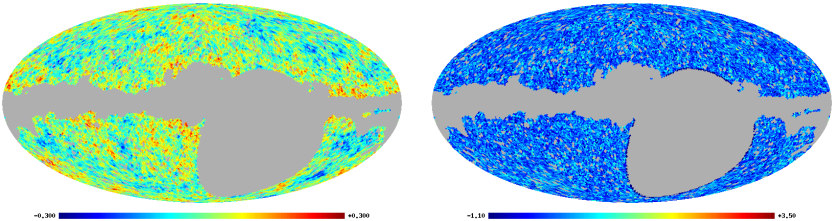
(a) No dark energy

(b) With dark energy

Figure: Analogy of ISW effect

Integrated Sachs Wolfe Effect

Correlation between CMB and LSS



(a) CMB (b) LSS

Figure: Constraining dark energy through any correlation between the CMB and LSS.

Recall wavelet transform in Euclidean space

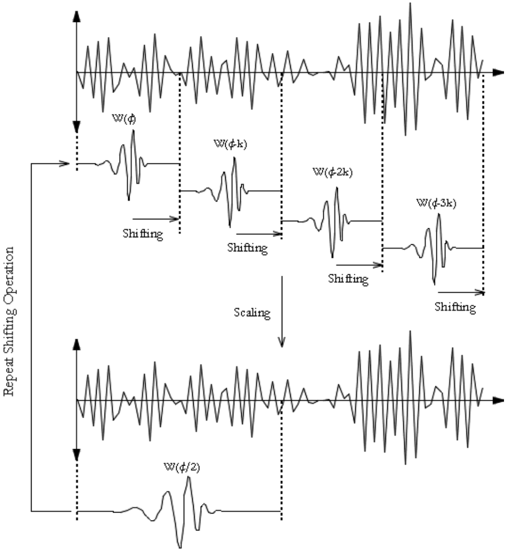


Figure: Wavelet scaling and shifting [Credit: `http://www.wavelet.org/tutorial/`]

Continuous wavelets on the sphere

- One of the first natural wavelet construction on the sphere was derived in the seminal work of Antoine and Vandergheynst (1998) (reintroduced by Wiaux 2005).

- Construct wavelet atoms from affine transformations (dilation, translation) on the sphere of a mother wavelet.

- The natural extension of translations to the sphere are rotations. Rotation of a function f on the sphere is defined by

$$\boxed{[\mathcal{R}(\rho)f](\omega) = f(\rho^{-1} \cdot \omega), \quad \omega = (\theta, \varphi) \in \mathbb{S}^2, \quad \rho = (\alpha, \beta, \gamma) \in \mathrm{SO}(3)\,.}$$

translation

- How define dilation on the sphere?

- The spherical dilation operator is defined through the conjugation of the Euclidean dilation and stereographic projection Π:

$$\boxed{\mathcal{D}(a) \equiv \Pi^{-1}\, d(a)\, \Pi\,.}$$

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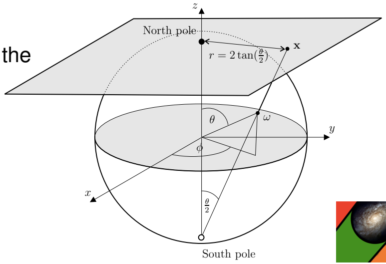

Continuous wavelets on the sphere

Forward transform (*i.e.* analysis)

- Wavelet family on the sphere constructed from rotations and dilations of a mother wavelet Ψ:

$$\boxed{\{\Psi_{a,\rho} \equiv \mathcal{R}(\rho)\mathcal{D}(a)\Psi : \rho \in \mathrm{SO}(3),\, a \in \mathbb{R}_*^+\}\,.}$$

dictionary

- The forward wavelet transform is given by

$$\boxed{W_\Psi^f(a,\rho) = \langle f, \Psi_{a,\rho}\rangle} = \int_{\mathbb{S}^2} \mathrm{d}\Omega(\omega)\, f(\omega)\, \Psi_{a,\rho}^*(\omega)\,,$$

projection

where $\mathrm{d}\Omega(\omega) = \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi$ is the usual invariant measure on the sphere.

- Wavelet coefficients live in $\mathrm{SO}(3) \times \mathbb{R}_*^+$; thus, directional structure is naturally incorporated.

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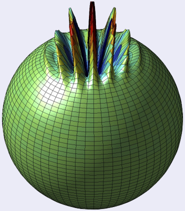
Continuous wavelets on the sphere

Fast algorithms

- Fast algorithms essential (for a review see Wiaux, McEwen & Vielva 2007)
  - Factoring of rotations: McEwen *et al.* (2007), Wandelt & Gorski (2001), Risbo (1996)
  - Separation of variables: Wiaux *et al.* (2005)

FastCSWT code http://www.fastcswt.org



Fast directional continuous spherical wavelet transform algorithms
McEwen *et al.* (2007)

- Fortran
- Supports directional and steerable wavelets

Continuous wavelets on the sphere

Mother wavelets

- Correspondence principle between spherical and Euclidean wavelets (Wiaux *et al.* 2005).
- Mother wavelets on sphere constructed from the projection of mother Euclidean wavelets defined on the plane:

$$\underbrace{\boxed{\Psi = \Pi^{-1}\Psi_{\mathbb{R}^2}}}_{\text{construction}},$$

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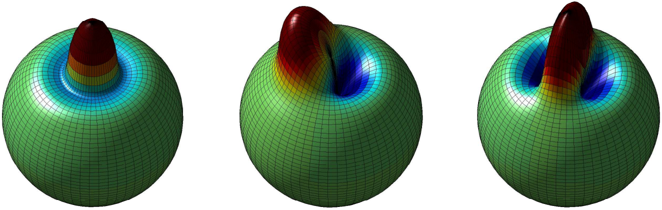

(a) SMHW (b) SBW (c) S2GDW

Figure: Spherical wavelets at scale $a, b = 0.2$.

Continuous wavelets on the sphere

Inverse transform (*i.e.* synthesis)

- The inverse wavelet transform given by

$$f(\omega) = \underbrace{\boxed{\int_0^\infty \frac{\mathrm{d}a}{a^3} \int_{\mathrm{SO}(3)} \mathrm{d}\varrho(\rho)}}_{\text{'sum' contributions}} \underbrace{\boxed{W_\Psi^f(a,\rho)\, [\mathcal{R}(\rho)\widehat{L}_\Psi \Psi_a](\omega)}}_{\text{weighted basis functions}} ,$$

  where $\mathrm{d}\varrho(\rho) = \sin\beta\, \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\gamma$ is the invariant measure on the rotation group $\mathrm{SO}(3)$.

- Perfect reconstruction iff wavelets satisfy admissibility property:

$$0 < \widehat{C}_\Psi^\ell \equiv \frac{8\pi^2}{2\ell+1} \sum_{m=-\ell}^{\ell} \int_0^\infty \frac{\mathrm{d}a}{a^3} \mid (\Psi_a)_{\ell m} \mid^2 < \infty, \quad \forall \ell \in \mathbb{N}$$

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- BUT... exact reconstruction not feasible in practice!

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Detecting dark energy

Wavelet coefficient correlation

- Compute wavelet correlation of CMB and LSS data (McEwen *et al.* 2007, McEwen *et al.* 2008).
- Compare to 1000 Monte Carlo simulations.
- Correlation detected at 99.9% significance.

$\Rightarrow$ Independent evidence for the existence of dark energy!

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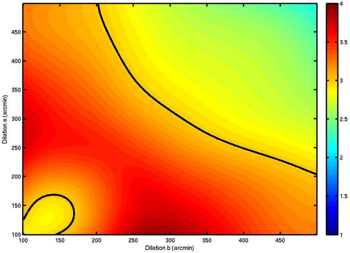

Figure: Wavelet correlation N_σ surface. Contours are shown at 3σ.

Detecting dark energy

Constraining cosmological models

- Use positive detection of the ISW effect to constrain parameters of cosmological models:
  - Energy density Ω_Λ.
  - Equation of state parameter w relating pressure and density of cosmological fluid modelling dark energy, *i.e.* $p = w\rho$.
- Parameter estimates of $\boxed{\Omega_\Lambda = 0.63^{+0.18}_{-0.17}}$ and $\boxed{w = -0.77^{+0.35}_{-0.36}}$ obtained.

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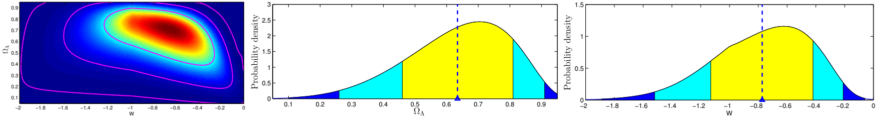

(a) Full likelihood surface (b) Marginalised distribution for Ω_Λ (c) Marginalised distribution for w

Figure: Likelihood for dark energy parameters.

Outline

Cosmic strings

- Symmetry breaking phase transitions in the early Universe $\rightarrow$ topological defects.
- Cosmic strings well-motivated phenomenon that arise when axial or cylindrical symmetry is broken $\rightarrow$ line-like discontinuities in the fabric of the Universe.
- We have not yet observed cosmic strings but we have observed string-like topological defects in other media.

The detection of cosmic strings would open a new window into the physics of the Universe!

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Observational signatures of cosmic strings

Conical Spacetime

- Spacetime about a cosmic string is conical, with a three-dimensional wedge removed (Vilenkin 1981).
- Strings moving transverse to the line of sight induce line-like discontinuities in the CMB (Kaiser & Stebbins 1984).
- The amplitude of the induced contribution scales with the string tension $G\mu$.

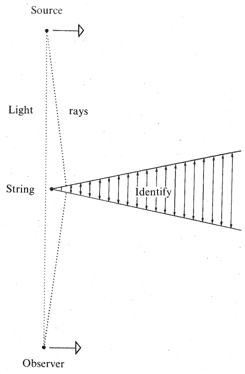


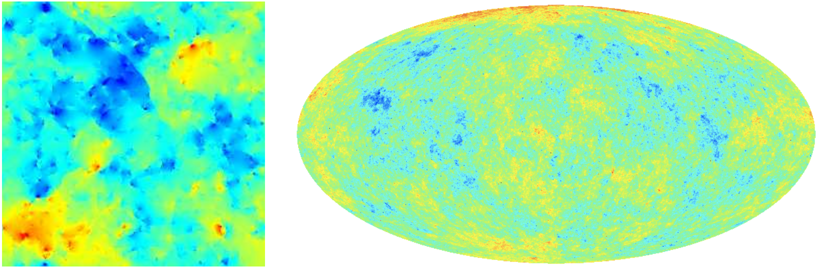
Figure: Spacetime around a cosmic string.
[Credit: Kaiser & Stebbins 1984, DAMTP.]

Observational signatures of cosmic strings

CMB contribution

- Make contact between theory and data using high-resolution simulations.
- Search for a weak string signal s embedded in the CMB c, with observations d given by

$$\underbrace{\boxed{d(\theta,\varphi)}}_{\text{observation}} = \underbrace{\boxed{c(\theta,\varphi)}}_{\text{CMB}} + \underbrace{\boxed{G\mu \cdot s(\theta,\varphi)}}_{\text{strings}} .$$



(a) Flat patch (Fraisse *et al.* 2008)   (b) Full-sky (Ringeval *et al.* 2012)

Figure: Cosmic string simulations.

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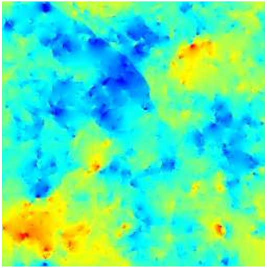

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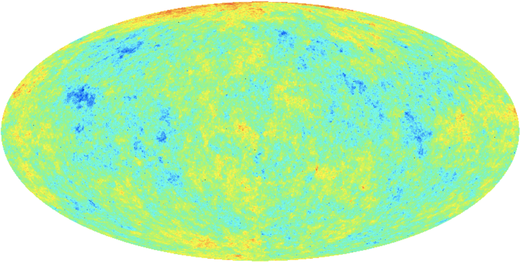

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Figure: Cosmic string simulations.

Scale-discretised wavelets on the sphere

Wavelet construction

Exact reconstruction not feasible in practice with continuous wavelets!

- *Exact reconstruction with directional wavelets on the sphere*
  Wiaux, McEwen, Vandergheynst, Blanc (2008)

- Dilation performed in harmonic space [*cf.* McEwen *et al.* (2006), Sanz *et al.* (2006)].

- Scale-discretised wavelet $\Psi^j \in L^2(\mathbb{S}^2, \mathrm{d}\Omega)$ defined in harmonic space:

$$\Psi^j_{\ell m} \equiv \kappa^j(\ell) s_{\ell m} \, .$$

- Admissible wavelets constructed to satisfy a resolution of the identity:

$$\underbrace{\boxed{|\Phi_{\ell 0}|^2}}_{\text{scaling function}} + \sum_{j=0}^{J} \sum_{m=-\ell}^{\ell} \underbrace{\boxed{|\Psi^j_{\ell m}|^2}}_{\text{wavelet}} = 1 \, , \quad \forall \ell \, .$$

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- Admissible wavelets constructed to satisfy a resolution of the identity:

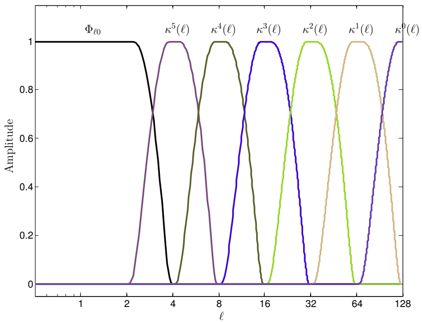
$$\underbrace{\boxed{|\Phi_{\ell 0}|^2}}_{\text{scaling function}} + \sum_{j=0}^{J} \sum_{m=-\ell}^{\ell} \underbrace{\boxed{|\Psi^j_{\ell m}|^2}}_{\text{wavelet}} = 1 \, , \quad \forall \ell \, .$$

Scale-discretised wavelets on the sphere

Wavelet construction

Exact reconstruction not feasible in practice with continuous wavelets!

- *Exact reconstruction with directional wavelets on the sphere*
  Wiaux, McEwen, Vandergheynst, Blanc (2008)
- Dilation performed in harmonic space [*cf.* McEwen *et al.* (2006), Sanz *et al.* (2006)].



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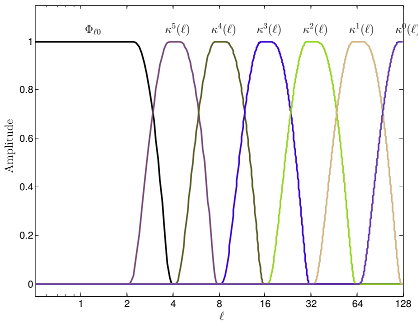

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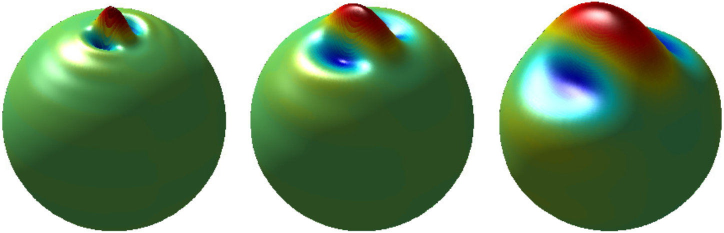
Scale-discretised wavelets on the sphere

Wavelets



(a) $j = 4$ (b) $j = 3$ (c) $j = 2$

Figure: Scale-discretised wavelets on the sphere.

Scale-discretised wavelets on the sphere

Forward and inverse transform (*i.e.* analysis and synthesis)

- The scale-discretised wavelet transform is given by the usual projection onto each wavelet:

$$\underbrace{\boxed{W^{\Psi^j}(\rho) = \langle f,\, \mathcal{R}_\rho \Psi^j \rangle}}_{\text{projection}} = \int_{\mathbb{S}^2} \mathrm{d}\Omega(\omega) f(\omega) (\mathcal{R}_\rho \Psi^j)^*(\omega)\ .$$

- The original function may be recovered exactly in practice from the wavelet (and scaling) coefficients:

$$f(\omega) = \underbrace{\boxed{2\pi \int_{\mathbb{S}^2} \mathrm{d}\Omega(\omega') W^{\Phi}(\omega') (\mathcal{R}_{\omega'} L^{\mathrm{d}} \Phi)(\omega)}}_{\text{scaling function contribution}} + \underbrace{\boxed{\sum_{j=0}^{J}}}_{\text{finite sum}} \underbrace{\boxed{\int_{\mathrm{SO}(3)} \mathrm{d}\varrho(\rho) W^{\Psi^j}(\rho) (\mathcal{R}_\rho L^{\mathrm{d}} \Psi^j)(\omega)\ .}}_{\text{wavelet contribution}}$$

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Scale-discretised wavelets on the sphere

Exact and efficient computation

- Wavelet analysis can be posed as an inverse Wigner transform on $\mathrm{SO}(3)$:

$$\boxed{W^{\Psi^j}(\rho) = \sum_{\ell=0}^{L-1} \sum_{m=-\ell}^{\ell} \sum_{n=-\ell}^{\ell} \frac{2\ell+1}{8\pi^2} \left(W^{\Psi^j}\right)_{mn}^{\ell} D_{mn}^{\ell *}(\rho) \, ,} \quad \text{where} \quad \left(W^{\Psi^j}\right)_{mn}^{\ell} = \frac{8\pi^2}{2\ell+1} f_{\ell m} \Psi_{\ell n}^{j*} \, .$$

which can be computed efficiently via a factoring of rotations (Risbo 1996, Wandelt & Gorski 2001, McEwen *et al.* 2007).

- Wavelet synthesis can be posed as an forward Wigner transform on $\mathrm{SO}(3)$:

$$f(\omega) \sim \sum_{j=0}^{J} \int_{\mathrm{SO}(3)} \mathrm{d}\varrho(\rho) W^{\Psi^j}(\rho) (\mathcal{R}_\rho L^{\mathrm{d}} \Psi^j)(\omega) = \sum_{j=0}^{J} \sum_{\ell m n} \frac{2\ell+1}{8\pi^2} \left(W^{\Psi^j}\right)_{mn}^{\ell} \Psi_{\ell n}^{j} Y_{\ell m}(\omega) \, ,$$

where

$$\boxed{\left(W^{\Psi^j}\right)_{mn}^{\ell} = \langle W^{\Psi^j}, D_{mn}^{\ell *} \rangle = \int_{\mathrm{SO}(3)} \mathrm{d}\varrho(\rho) W^{\Psi^j}(\rho) D_{mn}^{\ell}(\rho) \, ,}$$

which can be computed efficiently via a factoring of rotations (Risbo 1996, Wiaux, McEwen *et al.* 2008) and exactly by employing the Driscoll & Healy (1994) or McEwen & Wiaux (2011) sampling theorem.

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Scale-discretised wavelets on the sphere

Exact and efficient computation

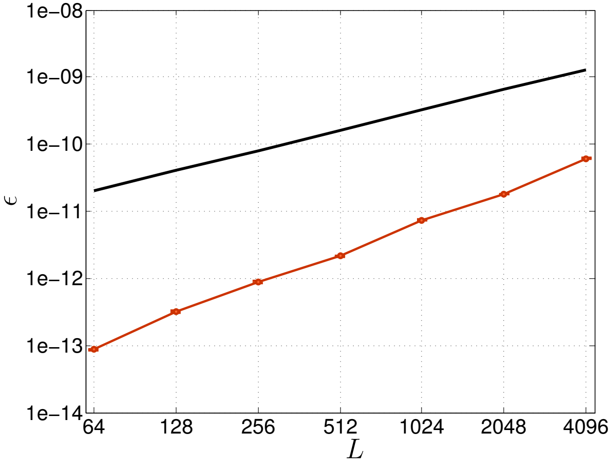


Figure: Numerical accuracy.

Scale-discretised wavelets on the sphere

Exact and efficient computation

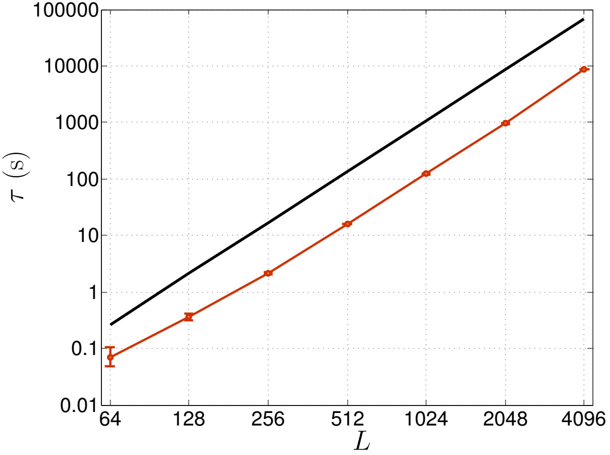


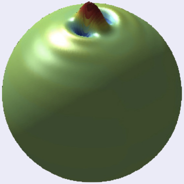
Figure: Computation time.

Scale-discretised wavelets on the sphere

Codes

S2DW code

http://www.s2dw.org

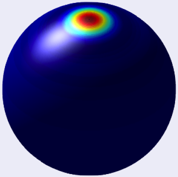


Exact reconstruction with directional wavelets on the sphere
Wiaux, McEwen, Vandergheynst, Blanc (2008)

- Fortran
- Parallelised
- Supports directional and steerable wavelets

S2LET code

http://www.s2let.org

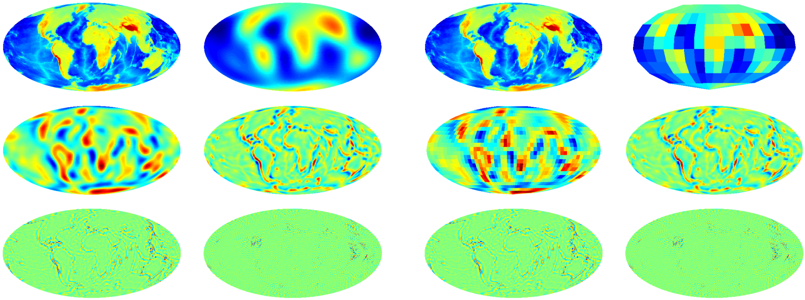


S2LET: A code to perform fast wavelet analysis on the sphere
Leistedt, McEwen, Vandergheynst, Wiaux (2012)

- C, Matlab, IDL, Java
- Supports only axisymmetric wavelets at present
- Future extensions planned (directional and steerable wavelets, faster algos, spin wavelets)

Scale-discretised wavelets on the sphere

Illustration



(a) Undecimated (b) Multi-resolution

Figure: Scale-discretised wavelet transform of a topography map of the Earth.

Motivation for using wavelets to detect cosmic strings

- Denote the wavelet coefficients of the data d by

$$\boxed{W^{d}_{j\rho} = \langle d,\ \Psi_{j\rho} \rangle}$$

for scale $j \in \mathbb{Z}^{+}$ and position $\rho \in \mathrm{SO}(3)$.

- Consider an even azimuthal band-limit $N = 4$ to yield wavelet with odd azimuthal symmetry.

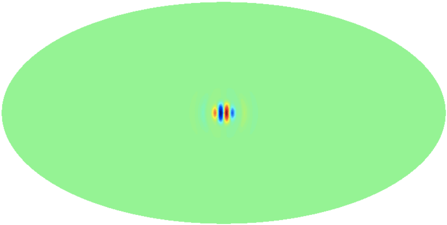


Figure: Example wavelet matched to the expected string contribution.

Motivation for using wavelets to detect cosmic strings

- Wavelet transform yields a sparse representation of the string signal → hope to effectively separate the CMB and string signal in wavelet space.

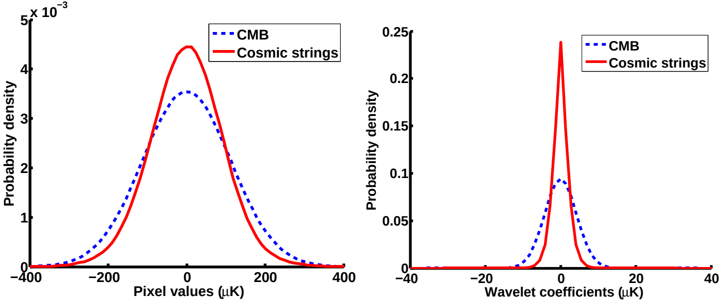


Figure: Distribution of CMB and string signal in pixel (left) and wavelet space (right).

Learning the statistics of the CMB and string signals in wavelet space

- Wavelet-Bayesian approach to estimate the string tension and map:

$$\underbrace{\boxed{d(\theta,\varphi)}}_{\text{observation}} = \underbrace{\boxed{c(\theta,\varphi)}}_{\text{CMB}} + \underbrace{\boxed{G\mu \cdot s(\theta,\varphi)}}_{\text{strings}} .$$

- Need to determine statistical description of the CMB and string signals in wavelet space.
- Calculate analytically the probability distribution of the CMB in wavelet space.
- Fit a generalised Gaussian distribution (GGD) for the wavelet coefficients of a string training map (*cf.* Wiaux *et al.* 2009):

$$\mathrm{P}_j^s(W_{j\rho}^s \mid G\mu) = \frac{\upsilon_j}{2G\mu\nu_j\Gamma(\upsilon_j{}^{-1})}\, \mathrm{e}^{\left(-\left|\frac{W_{j\rho}^s}{G\mu\nu_j}\right|^{\upsilon_j}\right)} ,$$

with scale parameter ν_j and shape parameter υ_j.

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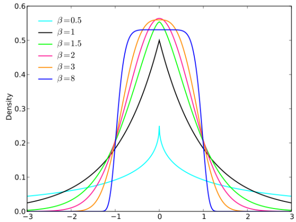
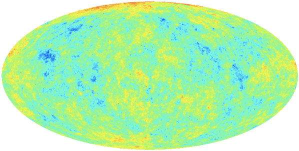
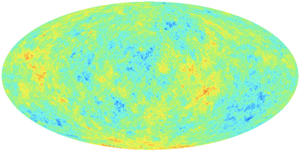


Figure: GGD

Learning the statistics of the CMB and string signals in wavelet space



(a) String training map



(b) String testing map

Figure: Cosmic string simulations.

- Distributions in close agreement.

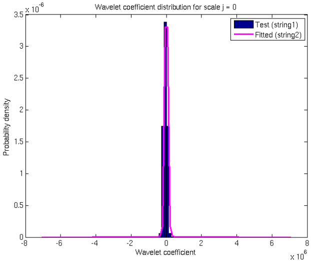
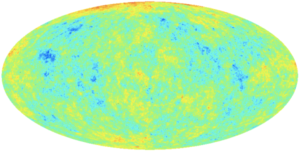


Figure: Distributions for wavelet scale $j = 0$.

Learning the statistics of the CMB and string signals in wavelet space



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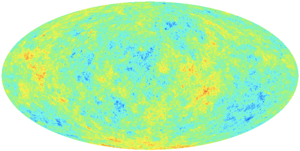

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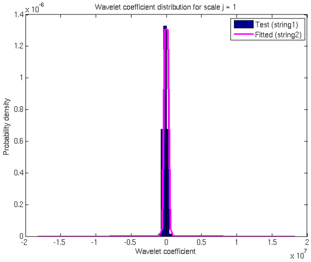
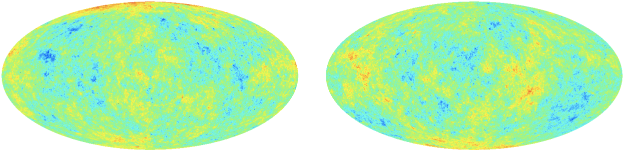


Figure: Distributions for wavelet scale $j = 1$.

Learning the statistics of the CMB and string signals in wavelet space



(a) String training map (b) String testing map

Figure: Cosmic string simulations.

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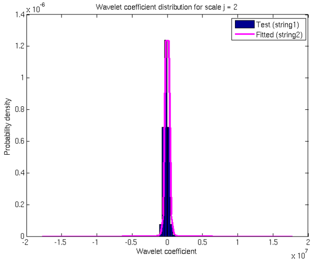
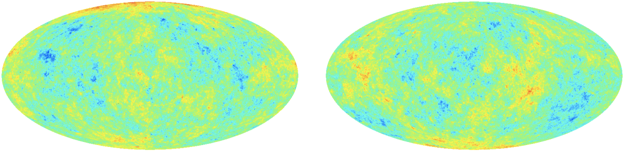


Figure: Distributions for wavelet scale $j = 2$.

Learning the statistics of the CMB and string signals in wavelet space



(a) String training map (b) String testing map

Figure: Cosmic string simulations.

- Distributions in close agreement.

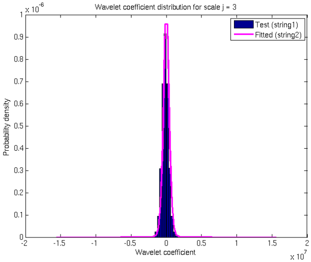
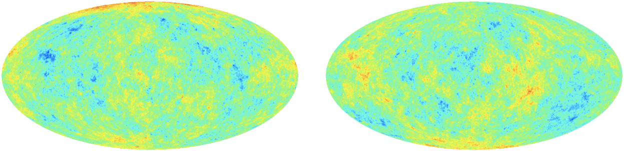


Figure: Distributions for wavelet scale $j = 3$.

Learning the statistics of the CMB and string signals in wavelet space



(a) String training map (b) String testing map

Figure: Cosmic string simulations.

- Distributions in close agreement.
- Accurately characterised statistics of string signals in wavelet space.

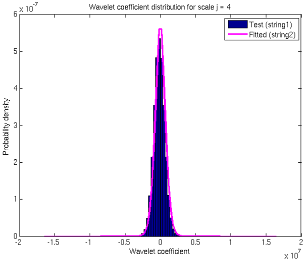


Figure: Distributions for wavelet scale $j = 4$.

Spherical wavelet-Bayesian string tension estimation

- Perform Bayesian string tension estimation in wavelet space.
- For each wavelet coefficient the likelihood is given by

$$\mathrm{P}(W^d_{j\rho} \,|\, G\mu) = \mathrm{P}(W^s_{j\rho} + W^c_{j\rho} \,|\, G\mu) = \int_{\mathbb{R}} \mathrm{d}W^s_{j\rho} \, \mathrm{P}^c_j(W^d_{j\rho} - W^s_{j\rho}) \, \mathrm{P}^s_j(W^s_{j\rho} \,|\, G\mu) \, .$$

- The overall likelihood of the data is given by

$$\mathrm{P}(W^d \,|\, G\mu) = \prod_{j,\rho} \mathrm{P}(W^d_{j\rho} \,|\, G\mu) \, ,$$

where we have assumed independence for numerical tractability.

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Spherical wavelet-Bayesian string tension estimation

- Compute the string tension posterior $\mathrm{P}(G\mu \mid W^d)$ by Bayes theorem:

$$\boxed{\mathrm{P}(G\mu \mid W^d) = \frac{\mathrm{P}(W^d \mid G\mu)\,\mathrm{P}(G\mu)}{\mathrm{P}(W^d)} \propto \mathrm{P}(W^d \mid G\mu)\,\mathrm{P}(G\mu)\ .}$$

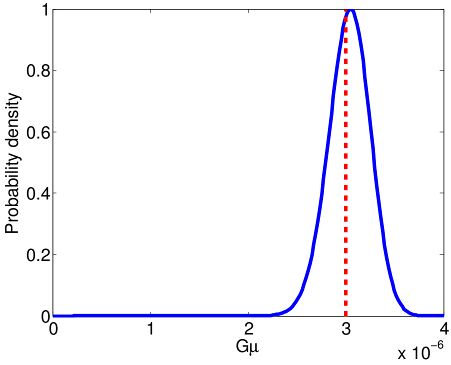

Figure: Posterior distribution of the string tension (true $G\mu = 3 \times 10^{-6}$).

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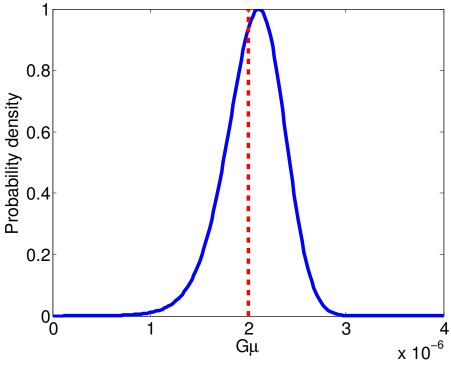

Figure: Posterior distribution of the string tension (true $G\mu = 2 \times 10^{-6}$).

Spherical wavelet-Bayesian string tension estimation

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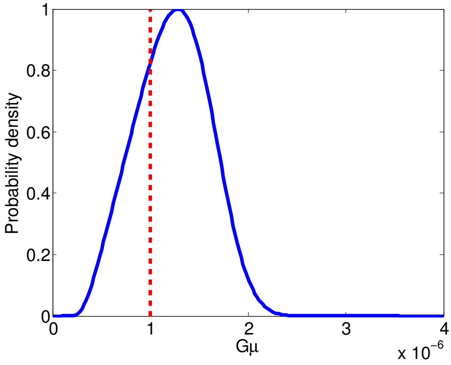

Figure: Posterior distribution of the string tension (true $G\mu = 1 \times 10^{-6}$).

Bayesian evidence for strings

- Compute Bayesian evidences to compare the string model M^s to the alternative model M^c that the observed data is comprised of just a CMB contribution.
- The Bayesian evidence of the string model is given by

$$E^s = \mathrm{P}(W^d \,|\, \mathrm{M}^s) = \int_{\mathbb{R}} \mathrm{d}(G\mu)\, \mathrm{P}(W^d \,|\, G\mu)\, \mathrm{P}(G\mu) \,.$$

- The Bayesian evidence of the CMB model is given by

$$E^c = \mathrm{P}(W^d \,|\, \mathrm{M}^c) = \prod_{j,\rho} \mathrm{P}_j^c(W_{j\rho}^d) \,.$$

- Compute the Bayes factor to determine the preferred model:

$$\Delta \ln E = \ln(E^s / E^c) \,.$$

Bayesian evidence for strings

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- The Bayesian evidence of the string model is given by

$$\boxed{E^s = \mathrm{P}(W^d \,|\, \mathrm{M}^s) = \int_{\mathbb{R}} \mathrm{d}(G\mu)\, \mathrm{P}(W^d \,|\, G\mu)\, \mathrm{P}(G\mu)\ .}$$

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$$\boxed{E^c = \mathrm{P}(W^d \,|\, \mathrm{M}^c) = \prod_{j,\rho} \mathrm{P}^c_j(W^d_{j\rho})\ .}$$

- Compute the Bayes factor to determine the preferred model:

$$\boxed{\Delta \ln E = \ln(E^s / E^c)\ .}$$

Bayesian evidence for strings

- Compute Bayesian evidences to compare the string model M^s to the alternative model M^c that the observed data is comprised of just a CMB contribution.
- The Bayesian evidence of the string model is given by

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Table: Tension estimates and log-evidence differences for simulations.

$G\mu/10^{-6}$	0.7	0.8	0.9	1.0	2.0	3.0
$\widehat{G\mu}/10^{-6}$	1.1	1.2	1.2	1.3	2.1	3.1
$\Delta\ln E$	-1.3	-1.1	-0.9	-0.7	5.5	29

Recovering string maps

- Inference of the wavelet coefficients of the underlying string map encoded in posterior probability distribution $\mathrm{P}(W^s_{j\rho} \mid W^d)$.
- Estimate the wavelet coefficients of the string map from the mean of the posterior distribution:

$$\overline{W}^s_{j\rho} = \int_{\mathbb{R}} \mathrm{d}W^s_{j\rho}\, W^s_{j\rho}\, \mathrm{P}(W^s_{j\rho} \mid W^d)$$

- Recover the string map from its wavelets (possible since the scale-discretised wavelet transform on the sphere supports exact reconstruction).
- Work in progress. . .

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Outline

Bianchi VII_h cosmologies

Test fundamental assumptions on which modern cosmology is based, *e.g.* isotropy.

- Relax assumptions about the global structure of spacetime by allowing anisotropy about each point in the universe, *i.e.* rotation and shear.
- Yields more general solutions to Einstein's field equations $\rightarrow$ Bianchi cosmologies.
- Induces a characteristic subdominant, deterministic signature in the CMB, which is embedded in the usual stochastic anisotropies (Collins & Hawking 1973, Barrow *et al.* 1985).

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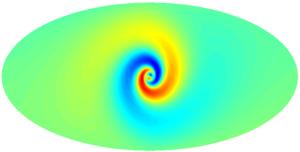

Figure: Bianchi CMB contribution.

Bianchi VII$_h$ cosmologies

Parameters

- Models described by the parameter vector:

$$\boxed{\Theta_{\mathrm{B}} = \left(\Omega_{\mathrm{m}},\, \Omega_{\Lambda},\, x,\, (\omega/H)_0,\, \alpha, \beta, \gamma\right) .}$$

- Free parameter, x, describing the comoving length-scale over which the principal axes of shear and rotation change orientation, *i.e.* ‘spiralness’.
- Amplitude characterised by the dimensionless vorticity $(\omega/H)_0$, which influences the amplitude of the induced temperature contribution only and not its morphology.
- The orientation and handedness of the coordinate system is also free.

Bianchi VII$_h$ cosmologies

Simulations

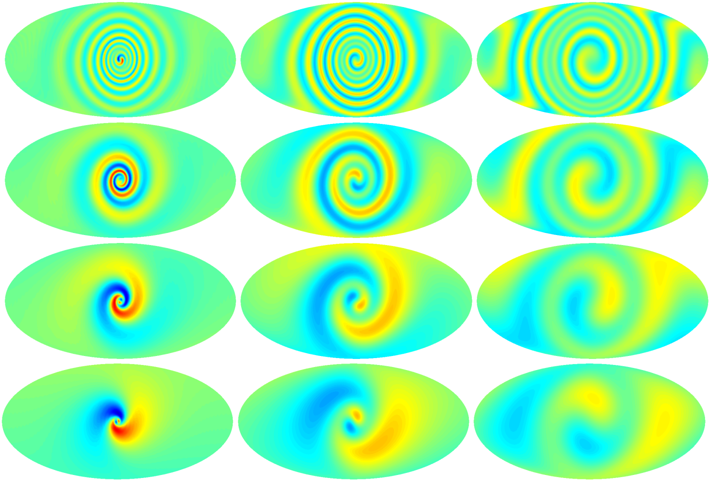

Figure: Simulated CMB contributions in Bianchi VII$_h$ cosmologies for varying parameters.

Bayesian analysis of Bianchi VII$_h$ cosmologies

Parameter estimation

- Perform Bayesian analysis of McEwen *et al.* (2013).
- Consider open and flat cosmologies with cosmological parameters:
  $\Theta_{\rm C} = (A_s,\, n_s,\, \tau,\, \Omega_{\rm b}h^2,\, \Omega_{\rm c}h^2,\, \Omega_\Lambda,\, \Omega_k)$.
- Recall Bianchi parameters:
  $\Theta_{\rm B} = \big(\Omega_{\rm m},\, \Omega_\Lambda,\, x,\, (\omega/H)_0,\, \alpha, \beta, \gamma\big)$.
- Likelihood given by

$$\mathrm{P}(\boldsymbol{d} \,|\, \Theta_{\rm B}, \Theta_{\rm C}) \propto \frac{1}{\sqrt{|\mathbf{X}(\Theta_{\rm C})|}} \mathrm{e}^{\left[-\chi^2(\Theta_{\rm C},\Theta_{\rm B})/2\right]} \,,$$

where

$$\chi^2(\Theta_{\rm C}, \Theta_{\rm B}) = \left[\boldsymbol{d} - \boldsymbol{b}(\Theta_{\rm B})\right]^\dagger \mathbf{X}^{-1}(\Theta_{\rm C}) \left[\boldsymbol{d} - \boldsymbol{b}(\Theta_{\rm B})\right] .$$

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Bayesian analysis of Bianchi VII$_h$ cosmologies

Covariance

- Bianchi VII$_h$ templates can be computed accurately and rotated efficiently in harmonic space $\rightarrow$ consider harmonic space representation, where $\boldsymbol{d} = \{d_{\ell m}\}$ and $\boldsymbol{b}(\Theta_{\rm B}) = \{b_{\ell m}(\Theta_{\rm B})\}$.
- Partial-sky analysis that handles in harmonic space a mask applied in pixel space.
- Add masking noise in order to marginalise the pixel values of the data contained in the masked region, with variance for pixel i given by $\sigma_m^2(\omega_i)$.
- The covariance is then given by

$$\mathbf{X}(\Theta_{\rm C}) = \mathbf{C}(\Theta_{\rm C}) + \mathbf{M}\,,$$

where

- $\mathbf{C}(\Theta_{\rm C})$ is the diagonal CMB covariance defined by the power spectrum $C_\ell(\Theta_{\rm C})$;
- $\mathbf{M}$ is the non-diagonal noisy mask covariance matrix defined by

$$\mathbf{M}_{\ell m}^{\ell' m'} = \langle m_{\ell m}\, m^*_{\ell' m'} \rangle \simeq \sum_{\omega_i} \sigma_m^2(\omega_i) Y^*_{\ell m}(\omega_i)\, Y_{\ell' m'}(\omega_i)\, {\Omega_{\rm pix}}^2\,.$$

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Bayesian analysis of Bianchi VII$_h$ cosmologies

Model selection

- Compute the Bayesian evidence to determine preferred model:

$$\boxed{E = \mathrm{P}(\boldsymbol{d} \,|\, M) = \int \mathrm{d}\Theta \, \mathrm{P}(\boldsymbol{d} \,|\, \Theta, M) \, \mathrm{P}(\Theta \,|\, M) \, .}$$

- Use MultiNest to compute the posteriors and evidences via nested sampling (Feroz & Hobson 2008, Feroz *et al.* 2009).

- Consider two models:
  - Flat-decoupled-Bianchi model: $\Theta_{\rm C}$ and $\Theta_{\rm B}$ fitted simultaneously but decoupled $\rightarrow$ phenomenological
  - Open-coupled-Bianchi model: $\Theta_{\rm C}$ and $\Theta_{\rm B}$ fitted simultaneously and coupled $\rightarrow$ physical

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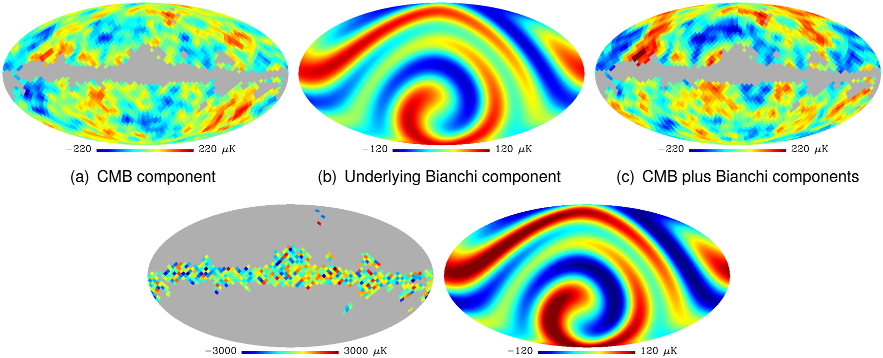
Bayesian analysis of Bianchi VII$_h$ cosmologies

Validation with simulations



(a) CMB component
(b) Underlying Bianchi component
(c) CMB plus Bianchi components
(d) Noisy mask realisation
(e) Recovered Bianchi component

Figure: Partial-sky simulation with embedded Bianchi VII$_h$ component at $L = 32$.

Bayesian analysis of Bianchi VII$_h$ cosmologies

Validation with simulations

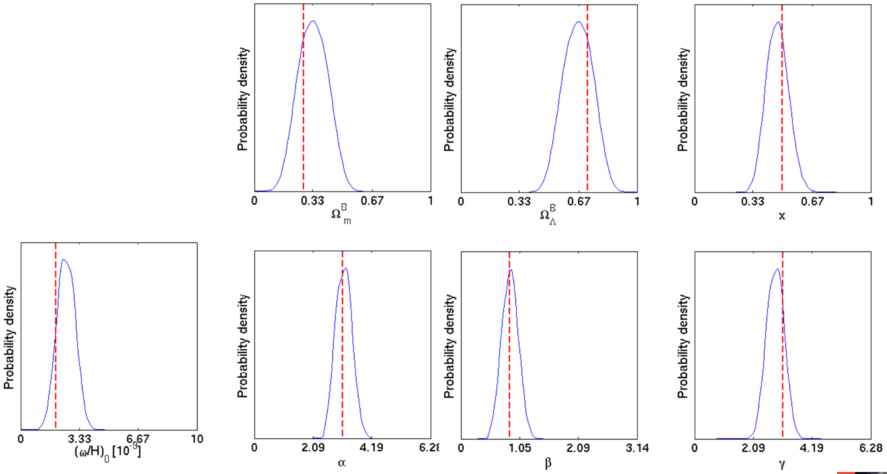


Figure: Marginalised posterior distributions recovered from partial-sky simulation at $L = 32$.

Planck results

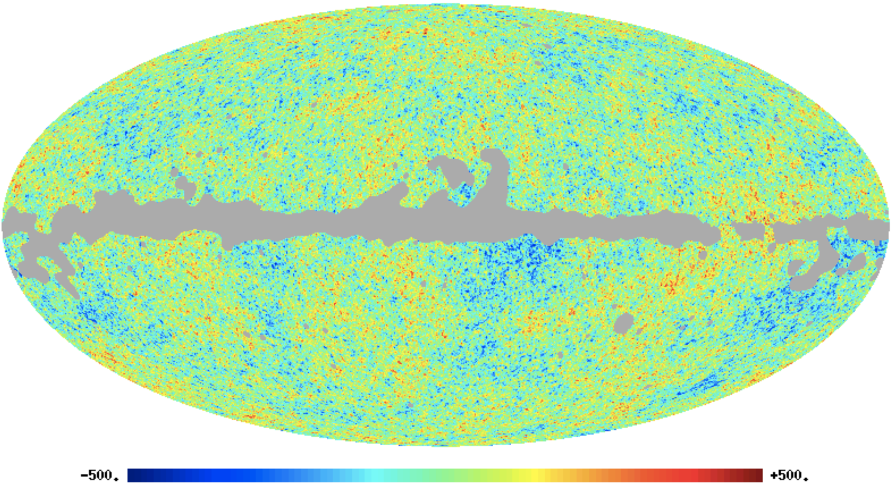


Figure: *Planck* SMICA component-separated data.

Planck results: flat-decoupled-Bianchi model

Parameter estimates

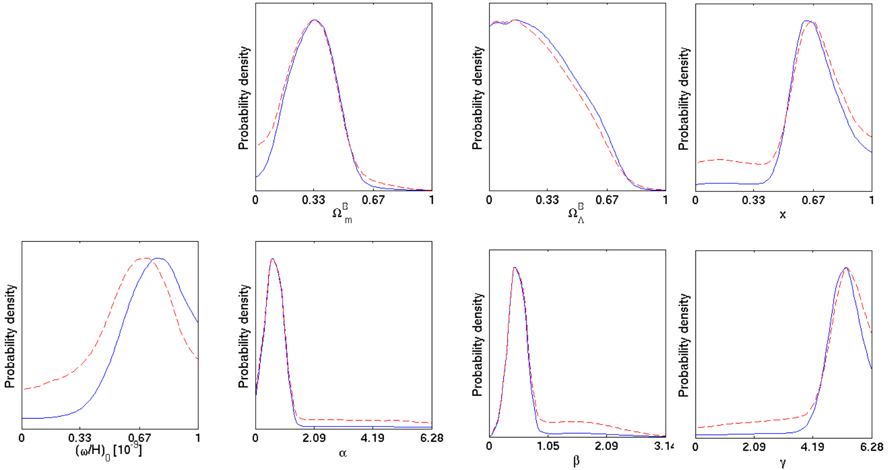


Figure: Posterior distributions of Bianchi parameters recovered for the phenomenological flat-decoupled-Bianchi model from *Planck* `SMICA` (solid curves) and `SEVEM` (dashed curves) data.

Planck results: flat-decoupled-Bianchi model

Bayesian evidence

Table: Bayes factor relative to equivalent ΛCDM model (positive favours Bianchi model).

Model	$\Delta \ln E$	
	SMICA	SEVEM
Flat-decoupled-Bianchi (left-handed)	2.8 ± 0.1	1.5 ± 0.1
Flat-decoupled-Bianchi (right-handed)	0.5 ± 0.1	0.5 ± 0.1

- On the Jeffreys (1961) scale, evidence for the inclusion of a Bianchi VII_h component would be termed strong (significant) for SMICA (SEVEM) component-separated data.
- A log-Bayes factor of 2.8 corresponds to an odds ratio of approximately 1 in 16.

Planck data favour the inclusion of a phenomenological Bianchi VII_h component!

Planck results: flat-decoupled-Bianchi model

Best-fit Bianchi component

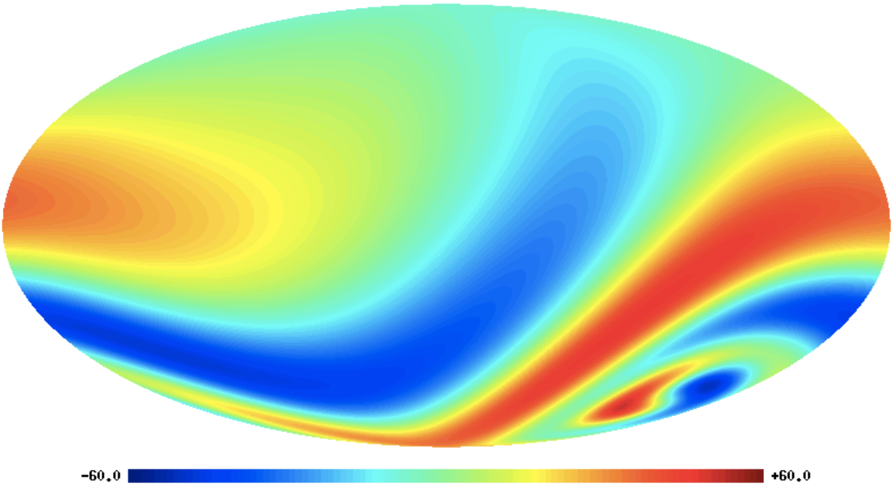


Figure: Best-fit template of flat-decoupled-Bianchi VII$_h$ model found in *Planck* `SMICA` component-separated data.

Planck results: flat-decoupled-Bianchi model

BUT the flat-Bianchi-decoupled model is phenomenological and **not physical!**

Parameter estimates are not consistent with concordance cosmology.

Planck results: open-coupled-Bianchi model

Bayesian evidence

Table: Bayes factor relative to equivalent ΛCDM model (positive favours Bianchi model).

Model	$\Delta \ln E$	
	SMICA	SEVEM
Open-coupled-Bianchi (left-handed)	0.0 ± 0.1	0.0 ± 0.1
Open-coupled-Bianchi (right-handed)	-0.4 ± 0.1	-0.4 ± 0.1

- In the physical setting where the standard cosmological and Bianchi parameters are coupled,

  Planck data do not favour the inclusion of a Bianchi VII$_h$ component.

- Find no evidence for Bianchi VII$_h$ cosmologies and constrain vorticity to:

$$(\omega/H)_0 < 8.1 \times 10^{-10}$$

95% confidence level

Planck results: open-coupled-Bianchi model

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