Multiscale Optimal Filtering on the Sphere

Adeem Aslam, *Student Member, IEEE*, Zubair Khalid, *Senior Member, IEEE* and Jason D. McEwen, *Senior Member, IEEE*

***Abstract*—We present a framework for the optimal filtering of spherical signals contaminated by realizations of an additive, zero-mean, uncorrelated and anisotropic noise process on the sphere. Filtering is performed in the wavelet domain given by the scale-discretized wavelet transform on the sphere. The proposed filter is optimal in the sense that it minimizes the mean square error between the filtered wavelet representation and wavelet representation of the noise-free signal. We also present a simplified formulation of the filter for the case when azimuthally symmetric wavelet functions are used. We demonstrate the use of the proposed optimal filter for denoising of an Earth topography map in the presence of additive, zero-mean, uncorrelated and white Gaussian noise, and show that the proposed filter performs better than the hard thresholding method and weighted spherical harmonic (weighted-SPHARM) signal estimation framework.**

***Index Terms*—2-sphere, scale-discretized wavelet transform, anisotropic process, spherical harmonics, bandlimited signals.**

I. Introduction

Signals exhibiting angular dependence are naturally defined on the surface of the sphere and are called spherical signals. Such signals arise in different fields of science and engineering, such as astronomy [1], [2], cosmology [3], [4], geodesy [5], computer graphics [6], [7], medical imaging [8], [9], acoustics [10], [11] and wireless communication [12], [13]. Signal acquisition in all of these application areas is influenced by unwanted, yet unavoidable, noise which places signal estimation from its noise-contaminated samples at the heart of signal processing techniques. There is an abundance of literature on signal filtering and estimation methods [14]–[22]. Some of these methods process signals using spatial (temporal) or spectral representation while others use joint spatial (temporal)-spectral representation.

Signal filtering and estimation has also been carried out using the joint scale-space representation in the Euclidean domain [23], [24]. Such a representation is enabled by the wavelet transform [25]–[27], which has proved invaluable in the multiscale analysis of one dimensional and multidimensional Euclidean domain signals. The framework for wavelet transform on the sphere has been extensively investigated in the literature as well [28]–[35]. Since dilation on the sphere, unlike in the Euclidean setting, can be defined in more than one way, there are different formulations and algorithms for multiscale analysis of spherical signals [31], [32], [34], [35].

Signal estimation using wavelet transform is based on the observation that noise has a distributed representation in the wavelet domain whereas signals of interest are typically sparsely represented, which can be exploited using different thresholding methods [36], [37]. Multiscale signal estimation in the Euclidean domain has also been carried out using signal and noise statistics through Wiener filtering [23]. In this work, we adopt the philosophy of Wiener filtering to propose a multiscale filter for signal estimation by using the scaled-discretized wavelet transform on the sphere [34], [35]. The designed filter is optimal in the sense that the filtered wavelet representation is the minimum mean square error estimate of the scale-discretized wavelet transform of the noise-free signal. Before presenting the design of the optimal filter, we review the fundamentals of signal analysis on the sphere and $\mathbb{SO}(3)$ rotation group in Section II. We briefly review the scale-discretized wavelet transform and present the optimal filtering framework in Section III. In Section IV, we demonstrate the performance of the proposed multiscale optimal filter by estimating a bandlimited Earth topography map in the presence of additive, zero-mean, uncorrelated and white Gaussian noise. We also test the robustness of the proposed multiscale filter, at different noise levels, against the hard thresholding method and the weighted spherical harmonic (weighted-SPHARM) signal estimation framework [8], before concluding in Section V.

II. Mathematical Preliminaries

A. Signals on 2-Sphere

Spherical domain, also referred to as 2-sphere (or just sphere) is defined as $\mathbb{S}^2 \triangleq \{\hat{\boldsymbol{x}} \in \mathbb{R}^3 : |\hat{\boldsymbol{x}}| = 1\}$, where $|\cdot|$ represents the Euclidean norm. A point $\hat{\boldsymbol{x}} \in \mathbb{S}^2$ can be represented as $\hat{\boldsymbol{x}}(\theta,\phi) \triangleq (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^{\mathrm{T}}$, where $(\cdot)^{\mathrm{T}}$ denotes vector transpose and the angles $\theta \in [0,\pi]$ and $\phi \in [0,2\pi)$ are referred to as colatitude and longitude, and are measured from the positive z-axis and positive x-axis (in the $x-y$ plane) respectively. Complex-valued and square-integrable functions defined on the sphere form a Hilbert space $L^2(\mathbb{S}^2)$, which is equipped with the following inner product for $f,g \in L^2(\mathbb{S}^2)$

$$\langle f, g\rangle_{\mathbb{S}^2} \triangleq \int_{\mathbb{S}^2} f(\hat{\boldsymbol{x}})\overline{g(\hat{\boldsymbol{x}})}ds(\hat{\boldsymbol{x}}), \quad \int_{\mathbb{S}^2} \triangleq \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}, \tag{1}$$

where $\overline{(\cdot)}$ denotes complex conjugate and $ds(\hat{\boldsymbol{x}}) = \sin\theta d\theta d\phi$. Norm of the function f is induced as $\|f\|_{\mathbb{S}^2} = \sqrt{\langle f, f\rangle_{\mathbb{S}^2}}$ and its energy[1] is given by $\langle f, f\rangle_{\mathbb{S}^2}$. For $L^2(\mathbb{S}^2)$, spherical harmonics, denoted by $Y_\ell^m(\hat{\boldsymbol{x}})$ for integer degree $\ell \geq 0$ and integer order $|m| \leq \ell$, form a complete set of orthonormal basis functions [38], which can be used to represent any signal $f \in L^2(\mathbb{S}^2)$ as

$$f(\hat{\boldsymbol{x}}) = \sum_{\ell,m}^{\infty} (f)_\ell^m Y_\ell^m(\hat{\boldsymbol{x}}), \quad \sum_{\ell,m}^{\infty} \equiv \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}, \tag{2}$$

[1]We refer to finite energy functions as signals.

A. Aslam and Z. Khalid are with the School of Science and Engineering, Lahore University of Management Science, Lahore, Pakistan (e-mail: adeem.aslam@lums.edu.pk, zubair.khalid@lums.edu.pk).

J. D. McEwen is with the Mullard Space Science Laboratory, University College London, Surrey, England (e-mail: jason.mcewen@ucl.ac.uk).

where $(f)^m_\ell \triangleq \langle f, Y^m_\ell \rangle_{\mathbb{S}^2}$ is the spherical harmonic (spectral) coefficient of degree ℓ and order m, and constitutes the spectral domain representation of the signal f. Signal f is considered bandlimited to degree L if $(f)^m_\ell = 0$ for $\ell \geq L, |m| \leq \ell$. Set of all such bandlimited signals forms an L^2-dimensional subspace of $L^2(\mathbb{S}^2)$, denoted by $\mathcal{H}_L$.

B. Signals on $\mathbb{SO}(3)$ Rotation Group

Rotations on the sphere are specified by three Euler angles namely, $\omega \in [0, 2\pi)$, $\vartheta \in [0, \pi]$ and $\varphi \in [0, 2\pi)$, using the right handed zyz convention, in which a point on the sphere is sequentially rotated by ω, ϑ and φ around z, y and z axes respectively. Group of all proper rotations[2], specified by the three Euler angles $(\varphi, \vartheta, \omega)$, is called the Special Orthogonal group, denoted by $\mathbb{SO}(3)$. Each element in the $\mathbb{SO}(3)$ rotation group is parameterized by the 3-tuple of Euler angles $\rho = (\varphi, \vartheta, \omega)$. We define rotation of signals on the sphere through an operator $\mathcal{D}_\rho$. Spectral representation of a signal $f \in L^2(\mathbb{S}^2)$ under the action of $\mathcal{D}_\rho$ is given by

$$(\mathcal{D}_\rho f)^m_\ell \triangleq \langle \mathcal{D}_\rho f, Y^m_\ell \rangle_{\mathbb{S}^2} = \sum_{m'=-\ell}^{\ell} D^\ell_{m,m'}(\rho)(f)^{m'}_\ell, \quad (3)$$

where $D^\ell_{m,m'}(\rho)$ is called the Wigner-D function of integer degree $\ell \geq 0$ and integer orders $|m|, |m'| \leq \ell$ [38].

For square-integrable and complex-valued functions, of the form $g(\rho), \nu(\rho)$, defined on the $\mathbb{SO}(3)$ rotation group, the inner product is given by

$$\langle g, \nu \rangle_{\mathbb{SO}(3)} \triangleq \int_{\mathbb{SO}(3)} g(\rho)\overline{\nu(\rho)}\, d\rho, \qquad \int_{\mathbb{SO}(3)} \triangleq \int_{\varphi=0}^{2\pi} \int_{\vartheta=0}^{\pi} \int_{\omega=0}^{2\pi}, \quad (4)$$

where $d\rho = d\varphi \sin\vartheta d\vartheta d\omega$. Equipped with the inner product in (4), the set of signals defined on the $\mathbb{SO}(3)$ rotation group form a Hilbert space, denoted by $L^2(\mathbb{SO}(3))$, which has Wigner-D functions as basis functions that satisfy the following orthogonality relationship [38]

$$\left\langle D^\ell_{m,m'}, D^p_{q,q'} \right\rangle_{\mathbb{SO}(3)} = c_\ell\, \delta_{\ell,p}\delta_{m,q}\delta_{m',q'},\ c_\ell \triangleq \frac{8\pi^2}{2\ell+1}. \quad (5)$$

Hence, we can express any signal $g \in L^2(\mathbb{SO}(3))$ as

$$g(\rho) = \sum_{\ell,m,m'}^{\infty} (g)^\ell_{m,m'} \overline{D^\ell_{m,m'}(\rho)}, \qquad \sum_{\ell,m,m'}^{\infty} \equiv \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{m'=-\ell}^{\ell}, \quad (6)$$

where $(g)^\ell_{m,m'} = (1/c_\ell) \left\langle g, \overline{D^\ell_{m,m'}} \right\rangle_{\mathbb{SO}(3)}$ forms the spectral domain representation of the signal g.

III. Optimal Filtering using Scale-Discretized Wavelet Transform

We consider a realization of an anisotropic random process as the source signal $s(\hat{\boldsymbol{x}})$, which is contaminated by a realization of an additive, zero-mean and anisotropic random process, called the noise signal $z(\hat{\boldsymbol{x}})$. The source and noise signals are assumed to be uncorrelated, i.e., $\mathbb{E}\{(s)^m_\ell \overline{(z)^q_p}\} = 0, \forall\, \ell, p, |m| \leq \ell, |q| \leq p$, where $\mathbb{E}\{\cdot\}$ represents the expectation operator. We further assume that the spectral covariances of the source and noise processes, denoted by $\mathbf{C}^s$, $\mathbf{C}^z$ respectively and defined as $C^s_{\ell m,pq} = \mathbb{E}\{(s)^m_\ell \overline{(s)^q_p}\}$, $C^z_{\ell m,pq} = \mathbb{E}\{(z)^m_\ell \overline{(z)^q_p}\}$, are known. Given the noise-contaminated observation $f(\hat{\boldsymbol{x}}) = s(\hat{\boldsymbol{x}}) + z(\hat{\boldsymbol{x}})$, the problem under consideration is to obtain an estimate of the source signal, denoted by $\tilde{s}(\hat{\boldsymbol{x}})$, by filtering the observation f at different scales using the scale-discretized wavelet transform. Before we design an optimal filter for signal estimation, we present a multiscale representation of signals in the wavelet domain.

A. Scale-Discretized Wavelet Transform on 2-Sphere

In this section, we briefly review the directional scale-discretized wavelet transform given in [34], [35]. For a signal $f \in L^2(\mathbb{S}^2)$, the directional scale-discretized wavelet transform is defined as

$$w^{\Psi^{(j)}}_f(\rho) \triangleq \left\langle f, \mathcal{D}_\rho \Psi^{(j)} \right\rangle_{\mathbb{S}^2} = \sum_{\ell,m,m'}^{\infty} (f)^m_\ell \overline{(\Psi^{(j)})^{m'}_\ell}\, \overline{D^\ell_{m,m'}(\rho)}, \quad (7)$$

where $w^{\Psi^{(j)}}_f \in L^2(\mathbb{SO}(3))$ is the wavelet coefficient of the signal f, j is the discrete wavelet scale, $(\Psi^{(j)})^{m'}_\ell$ is the spectral coefficient of the wavelet function $\Psi^{(j)} \in L^2(\mathbb{S}^2)$ and we have used (3) along with the orthonormality of spherical harmonics to evaluate the second equality. Wavelet functions capture the high frequency (rapid variations along the space) content of the signal. Low frequency content of the signal is represented by the scaling coefficient, defined as

$$w^\Phi_f(\hat{\boldsymbol{x}}) = \langle f, \mathcal{D}_{\hat{\boldsymbol{x}}} \Phi \rangle_{\mathbb{S}^2} = \sum_{\ell,m}^{\infty} \sqrt{\frac{c_\ell}{2\pi}} (f)^m_\ell \overline{(\Phi)^0_\ell}\ Y^m_\ell(\hat{\boldsymbol{x}}), \quad (8)$$

for the azimuthally symmetric scaling function $\Phi \in L^2(\mathbb{S}^2)$[3], where $\mathcal{D}_{\hat{\boldsymbol{x}}} \triangleq \mathcal{D}(\varphi, \vartheta, 0)$ and we have used the following relation between Wigner-D functions and spherical harmonics [38]

$$D^\ell_{m,0}(\varphi, \vartheta, 0) = \sqrt{\frac{4\pi}{2\ell+1}} \overline{Y^m_\ell(\vartheta, \varphi)}. \quad (9)$$

Signal f can be reconstructed perfectly from its wavelet and scaling coefficients using the following expression

$$f(\hat{\boldsymbol{x}}) = \int_{\mathbb{S}^2} w^\Phi_f(\hat{\boldsymbol{y}})(\mathcal{D}_{\hat{\boldsymbol{y}}}\Phi)(\hat{\boldsymbol{x}}) ds(\hat{\boldsymbol{y}}) + \int_{\mathbb{SO}(3)} \sum_{j=j_1}^{j_2} w^{\Psi^{(j)}}_f(\rho)(\mathcal{D}_\rho \Psi^{(j)})(\hat{\boldsymbol{x}}) d\rho, \quad (10)$$

provided the following admissibility condition holds

$$c_\ell \left(\frac{1}{2\pi} \left|(\Phi)^0_\ell\right|^2 + \sum_{j=j_1}^{j_2} \sum_{m'=-\ell}^{\ell} \left| \left(\Psi^{(j)}\right)^{m'}_\ell \right|^2 \right) = 1,\ \forall\ \ell, \quad (11)$$

where j_1 and j_2 denote the minimum and maximum wavelet scale[4] respectively and this condition can be derived from (10) using (7) and (8). Wavelet and scaling functions, which satisfy the admissibility condition, have the following spectral representation

$$\left(\Psi^{(j)}\right)^m_\ell = \frac{1}{\sqrt{c_\ell}} \Gamma^{(j)}_\Psi(\ell, \gamma)\, \zeta^m_\ell,\ (\Phi)^0_\ell = \sqrt{\frac{2\pi}{c_\ell}}\, \Gamma_\Phi(\ell, \gamma), \quad (12)$$

such that $\sum_{m=-\ell}^{\ell} |\zeta^m_\ell|^2 = 1$ for all values of ℓ for which ζ^m_ℓ is non-zero for at least one value of m, and $|\Gamma_\Phi|^2 +$

[2]An improper rotation is a reflection about either an axis or the center of the spherical coordinate system.

[3]For an azimuthally symmetric function f, $(f)^m_\ell = 0, \forall m \neq 0$.

[4]Wavelet scale is inversely proportional to the degree ℓ.

$\sum_{j=j_1}^{j_2} \left|\Gamma_{\Psi}^{(j)}\right|^2 = 1$. Here γ is the harmonic space dilation parameter, $\Gamma_{\Psi}^{(j)}$, Γ_{Φ} are the harmonic tiling functions which control the angular localization, and ζ_{ℓ}^{m} encodes the directional features of the wavelet functions.[5]

B. Optimal Filtering in the Wavelet Domain

We define a joint $\mathbb{SO}(3)$-scale domain filter function as

$$\Xi(\rho; j) = \sum_{\ell,m,m'}^{L_{\Xi_j}-1} \left(\Xi(\cdot;j)\right)_{m,m'}^{\ell} \overline{D_{m,m'}^{\ell}(\rho)}, \tag{13}$$

for wavelet scales $j = j_1, \dots, j_2$. Action of this filter on wavelet coefficients of the noise-contaminated observation f is given by $\mathbb{SO}(3)$ convolution, defined as [39], [40]

$$w_{\tilde{s}}^{\Psi^{(j)}}(\rho) = \left(\Xi(\cdot;j) \circledast w_{f}^{\Psi^{(j)}}\right)(\rho) = \int_{\mathbb{SO}(3)} \Xi(\rho\rho_1^{-1}; j) w_{f}^{\Psi^{(j)}}(\rho_1) d\rho_1,$$

where $\tilde{s}(\hat{\boldsymbol{x}})$ is the source signal estimate and $\circledast$ denotes convolution on the $\mathbb{SO}(3)$ rotation group. Using the addition formula for Wigner-D functions [41], $\mathbb{SO}(3)$ convolution can be written in a computationally more amenable form as [40]

$$w_{\tilde{s}}^{\Psi^{(j)}}(\rho) = \sum_{\ell,m,m'}^{L-1} c_{\ell} \sum_{k=-\ell}^{\ell} (f)_{\ell}^{k} \left(\Xi(\cdot;j)\right)_{m,k}^{\ell} \overline{(\Psi^{(j)})_{\ell}^{m'} D_{m,m'}^{\ell}(\rho)}, \tag{14}$$

where c_{ℓ} is given in (5), L is the bandlimit of signal f and we have assumed, without loss of generality, that bandlimit of the filter function at each scale is equal to the signal bandlimit, i.e., $L_{\Xi_j} = L$. We design an optimal filter that minimizes the joint $\mathbb{SO}(3)$-scale domain mean square error given by

$$\mathcal{E}_{ms} = \mathbb{E}\left\{ \sum_{j=j_1}^{j_2} \| w_{\tilde{s}}^{\Psi^{(j)}}(\rho) - w_{s}^{\Psi^{(j)}}(\rho) \|_{\mathbb{SO}(3)}^2 \right\}, \tag{15}$$

and present the result in the following theorem.

Theorem 1. *Let the source signal $s(\hat{\boldsymbol{x}})$ be a realization of an anisotropic process which is contaminated by a realization of an additive, zero-mean and anisotropic noise process, $z(\hat{\boldsymbol{x}})$, to obtain the observation $f(\hat{\boldsymbol{x}}) = s(\hat{\boldsymbol{x}}) + z(\hat{\boldsymbol{x}})$. The source and noise signals are uncorrelated, i.e., $\mathbb{E}\{(s)_{\ell}^{m}\overline{(z)_{p}^{q}}\} = 0, \forall\, \ell, p, |m| \le \ell, |q| \le p$, with known spectral covariance matrices, defined as $C_{\ell m,pq}^{s} = \mathbb{E}\{(s)_{\ell}^{m}\overline{(s)_{p}^{q}}\}$ and $C_{\ell m,pq}^{z} = \mathbb{E}\{(z)_{\ell}^{m}\overline{(z)_{p}^{q}}\}$. Defining c_{ℓ} as in (5), spectral coefficients of the joint $\mathbb{SO}(3)$-scale domain filter in (13), which minimize the joint $\mathbb{SO}(3)$-scale domain mean square error in (15), can be obtained by inverting the following linear system*

$$\mathbf{A}^{\mathrm{T}}(\ell)\, \boldsymbol{\Xi}(j,\ell,m) = \mathbf{b}(\ell,m), \tag{16}$$

for $|m| \le \ell, 0 \le \ell \le L-1, j_1 \le j \le j_2$, where elements of the column vector $\boldsymbol{\Xi}(j,\ell,m)$ are given $\Xi_k = \left(\Xi(\cdot;j)\right)_{m,k}^{\ell}, |k| \le \ell$, and elements of the matrix $\mathbf{A}$ and column vector $\mathbf{b}$ are given by

$$A_{k,k'} = c_{\ell} \sum_{k=-\ell}^{\ell} \left(C_{\ell k,\ell k'}^{s} + C_{\ell k,\ell k'}^{z} \right), \; |k|, |k'| \le \ell, \tag{17}$$

$$b_{k'} = C_{\ell m,\ell k'}^{s}, \; |k'| \le \ell. \tag{18}$$

[5]See [34], [35] for details on the construction of harmonic tiling functions and directionality component of the wavelet functions.

Proof. Using (7) and (14), along with the orthogonality of Wigner-D functions on the $\mathbb{SO}(3)$ rotation group, the mean square error in (15) can be written as

$$\mathcal{E}_{ms} = \sum_{j=j_1}^{j_2} \sum_{\ell,m,m'}^{L-1} c_{\ell} \times$$
$$\mathbb{E}\Bigg\{ \left(c_{\ell} \sum_{k=-\ell}^{\ell} (f)_{\ell}^{k} \overline{(\Psi^{(j)})_{\ell}^{m'}} \left(\Xi(\cdot;j)\right)_{m,k}^{\ell} - (s)_{\ell}^{m} \overline{(\Psi^{(j)})_{\ell}^{m'}} \right) \times$$
$$\overline{\left(c_{\ell} \sum_{k'=-\ell}^{\ell} (f)_{\ell}^{k'} \overline{(\Psi^{(j)})_{\ell}^{m'}} \left(\Xi(\cdot;j)\right)_{m,k'}^{\ell} - (s)_{\ell}^{m} \overline{(\Psi^{(j)})_{\ell}^{m'}} \right)} \Bigg\}.$$

Differentiating $\mathcal{E}_{ms}$ with respect to $\overline{\left(\Xi(\cdot;j)\right)_{m,k'}^{\ell}}$ and setting the result equal to zero, we obtain a linear system which, using (17) and (18), can be cast in the matrix form in (16). $\square$

Having found the spectral representation of the joint $\mathbb{SO}(3)$-scale domain filter, signal estimate $\tilde{s}(\hat{\boldsymbol{x}})$ is obtained from the wavelet coefficients in (14) using (10).

Remark 1. *For wavelet functions which are azimuthally symmetric, i.e., $\left(\Psi^{(j)}\right)_{\ell}^{m} = \left(\Psi^{(j)}\right)_{\ell}^{0} \delta_{m,0}$, wavelet coefficients are defined on the sphere as [37]*

$$w_{f}^{\Psi^{(j)}}(\hat{\boldsymbol{x}}) = \left\langle f, \mathcal{D}_{\hat{\boldsymbol{x}}} \Psi^{(j)} \right\rangle_{\mathbb{S}^2} = \sum_{\ell,m}^{\infty} \sqrt{\frac{c_{\ell}}{2\pi}} (f)_{\ell}^{m} \overline{(\Psi^{(j)})_{\ell}^{0}}\, Y_{\ell}^{m}(\hat{\boldsymbol{x}}).$$

Hence, instead of the joint $\mathbb{SO}(3)$-scale domain filter, we design a joint spatial-scale domain filter as

$$\Xi(\hat{\boldsymbol{x}}; j) = \sum_{\ell,m}^{L_{\Xi_j}-1} \left(\Xi(\cdot;j)\right)_{\ell}^{m} Y_{\ell}^{m}(\hat{\boldsymbol{x}}), \tag{19}$$

whose action on the noise-contaminated observation f is given by the following $\mathbb{S}^2$ convolution [42]

$$w_{\tilde{s}}^{\Psi^{(j)}}(\hat{\boldsymbol{x}}) = \sum_{\ell,m}^{L-1} \left(\sqrt{\frac{c_{\ell}}{2\pi}} (f)_{\ell}^{m} \overline{(\Psi^{(j)})_{\ell}^{0}} \right) \left(\Xi(\cdot;j)\right)_{\ell}^{m} Y_{\ell}^{m}(\hat{\boldsymbol{x}}), \tag{20}$$

which necessitates the minimization of the following joint spatial-scale domain mean square error for finding the spectral coefficients of the filter function in (19)

$$\mathcal{E}_{ms} = \mathbb{E}\left\{ \sum_{j=j_1}^{j_2} \| w_{\tilde{s}}^{\Psi^{(j)}}(\hat{\boldsymbol{x}}) - w_{s}^{\Psi^{(j)}}(\hat{\boldsymbol{x}}) \|_{\mathbb{S}^2}^2 \right\}. \tag{21}$$

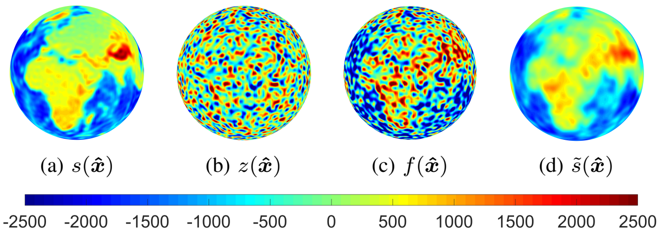
From the relation between Wigner-D functions and spherical harmonics in (9), it can be seen that (20) can be obtained from (14) by setting $m' = 0$ and $\left(\Xi(\cdot;j)\right)_{m,k}^{\ell} = (1/c_{\ell}) \left(\Xi(\cdot;j)\right)_{\ell}^{m} \delta_{m,k}$. Hence, by setting $k = k' = m$ in (17) and (18), spectral coefficients of the filter in (19) can be directly obtained from (16) as

$$\left(\Xi(\cdot;j)\right)_{\ell}^{m} = \frac{C_{\ell m,\ell m}^{s}}{(C_{\ell m,\ell m}^{s} + C_{\ell m,\ell m}^{z})}. \tag{22}$$

IV. Analysis

Performance of the multiscale optimal filter using scale-discretized wavelet transform is gauged by the signal-to-noise ratio, which for any signal $d \in L^2(\mathbb{S}^2)$ is defined as

$$\mathrm{SNR}^{d} = 20 \log \frac{\|s(\hat{\boldsymbol{x}})\|_{\mathbb{S}^2}}{\|d(\hat{\boldsymbol{x}}) - s(\hat{\boldsymbol{x}})\|_{\mathbb{S}^2}}. \tag{23}$$



(a) $s(\hat{\boldsymbol{x}})$ (b) $z(\hat{\boldsymbol{x}})$ (c) $f(\hat{\boldsymbol{x}})$ (d) $\tilde{s}(\hat{\boldsymbol{x}})$

Fig. 1: Multiscale optimal filtering of the bandlimited Earth topography map $s(\hat{\boldsymbol{x}})$, which is contaminated with zero-mean, uncorrelated and white Gaussian noise $z(\hat{\boldsymbol{x}})$ at $\mathrm{SNR}^f = -0.057$ dBs. Output SNR obtained from the source signal estimate $\tilde{s}(\hat{\boldsymbol{x}})$ is 9.68 dBs, resulting in SNR gain of 9.7 dBs.

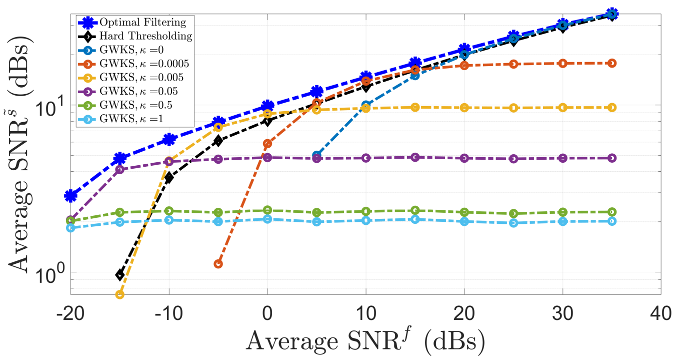


Fig. 2: Output SNR plotted against input SNR, averaged over 10 realizations of zero-mean, uncorrelated and white Gaussian noise process, for the proposed filtering framework (blue curve), hard thresholding method (black curve), and weighted-SPHARM based GWKS at different values of κ.

From this definition, input and output SNRs are given by SNR^f and $\mathrm{SNR}^{\tilde{s}}$ respectively. We demonstrate the utility of the multiscale optimal filter by using the Earth topography map[6], bandlimited to degree $L = 64$, as the source signal $s(\hat{\boldsymbol{x}})$, and contaminating it with realizations of zero-mean, uncorrelated and white Gaussian noise process at different values of SNR^f. Spectral covariance matrix of the source signal is computed as $C^s_{\ell m,pq} = (s)^m_\ell \overline{(s)^q_p}$. Spectral covariance matrix of the Gaussian noise process is generated as $\mathbf{C}^z = \sigma^2\mathbf{I}$, where $\mathbf{I}$ is identity matrix and σ^2 is the normalized noise energy, i.e.,

$$\sigma^2 = \frac{1}{L^2} 10^{-\mathrm{SNR}^f/10} \sum_{\ell,m}^{L-1} |(s)^m_\ell|^2 . \tag{24}$$

We use azimuthally symmetric wavelet and scaling functions for filtering and estimation of the source signal using the scale-discretized wavelet transform. At $L = 64$, the maximum wavelet scale $j_2 = 6$ [35], and we choose the minimum wavelet scale $j_1 = 0$. Dilation parameter for the harmonic tiling functions is set to 2 [35]. Fig. 1 shows an illustration of the optimal filtering framework in which the bandlimited Earth topography map is contaminated with zero-mean, uncorrelated and white Gaussian noise at $\mathrm{SNR}^f = -0.057$ dBs. Output SNR is measured to be 9.68 dBs, resulting in a significant SNR gain of 9.7 dBs.

We compare the performance of the proposed filtering framework with the hard thresholding method for signal denoising [37], by setting the threshold equal to $3\sigma_j$, where $\sigma_j^2 \triangleq \mathbb{E}\left\{\left|w_z^{\Psi^{(j)}}(\hat{\boldsymbol{x}})\right|^2\right\}$ is the noise variance in the wavelet domain at scale j and is given by [37]

$$\sigma_j^2 = \sigma^2 \sum_{\ell=0}^{L-1} \left|\left(\Psi^{(j)}\right)^0_\ell\right|^2, \quad \sigma^2 = \frac{\mathrm{trace}(\mathbf{C}^z)}{L^2}. \tag{25}$$

We also compare the proposed multiscale optimal filter with the weighted-SPHARM framework presented in [8], which uses Gauss-Weierstrass kernel smoothing (GWKS) to obtain the signal estimate by minimizing the following weighted squared error[7]

$$\int_{\mathbb{S}^2}\int_{\mathbb{S}^2} \overline{K_{GW}(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}})}\, |f(\hat{\boldsymbol{y}}) - \tilde{s}(\hat{\boldsymbol{x}})|^2\, ds(\hat{\boldsymbol{y}}) ds(\hat{\boldsymbol{x}}), \tag{26}$$

where $K_{GW}(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}})$ is the self-adjoint Gauss-Weierstrass (GW) kernel given by [43]

$$K_{GW}(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}}) = \sum_{\ell,m}^{L-1} e^{-\ell(\ell+1)\kappa} Y^m_\ell(\hat{\boldsymbol{x}}) \overline{Y^m_\ell(\hat{\boldsymbol{y}})},\ \kappa \in [0,1]. \tag{27}$$

The signal estimate obtained from the weighted-SPHARM framework is given by [8]

$$\tilde{s}(\hat{\boldsymbol{x}}) = \sum_{\ell,m}^{L-1} e^{-\ell(\ell+1)\kappa} (f)^m_\ell Y^m_\ell(\hat{\boldsymbol{x}}). \tag{28}$$

We contaminate the Earth topography map, bandlimited to degree $L = 64$, with 10 realizations of zero-mean, uncorrelated and white Gaussian noise process at different values of SNR^f, and compute $\mathrm{SNR}^{\tilde{s}}$. Fig. 2 shows the output SNR versus input SNR, averaged over 10 realizations[8], in which the proposed multiscale filtering method can be seen to perform better than the hard thresholding method, particularly in the low SNR regime. Since there is no systematic way of choosing the GW kernel parameter κ for weighted-SPHARM, we estimate the signal at various values of κ in the interval $[0, 1]$, and show in Fig. 2 that the proposed multiscale filter outperforms the weighted-SPHARM based GWKS at all values of κ.

V. Conclusion

We have formulated an optimal filter in the wavelet domain, using the scale-discretized wavelet transform, for estimation of signals contaminated by realizations of an additive, zero-mean, uncorrelated and anisotropic noise process on the sphere. The designed filter is optimal in the sense that the filtered representation in the wavelet domain is the minimum mean square error estimate of the wavelet representation of the noise-free signal. We have illustrated the utility of the proposed filtering framework on a bandlimited Earth topography map in the presence of additive, zero-mean, uncorrelated and white Gaussian noise, and have compared the proposed multiscale optimal filtering framework with the hard thresholding and weighted-SPHARM based methods for signal estimation, showing the proposed filter to be superior in performance at different noise levels.

[6]http://geoweb.princeton.edu/people/simons/software.html

[7]We use conjugate of the kernel function in (26) so as to reproduce the results of weighted-SPHARM in [8] using complex spherical harmonics.

[8]Average output SNR becomes negative in Fig. 2 (and hence, is not shown on the logarithmic scale) at some values of the input SNR for the hard thresholding and weighted-SPHARM methods.

REFERENCES

[1] N. Jarosik, C. L. Bennett, J. Dunkley, B. Gold, M. R. Greason, M. Halpern, R. S. Hill, G. Hinshaw, A. Kogut, E. Komatsu, D. Larson, M. Limon, S. S. Meyer, M. R. Nolta, N. Odegard, L. Page, K. M. Smith, D. N. Spergel, G. S. Tucker, J. L. Weiland, E. Wollack, and E. L. Wright, "Seven-year Wilkinson Microwave Anisotropy Probe (WMAP) observations: Sky maps, systematic errors, and basic results," *Astrophys. J.*, vol. 192, no. 2, pp. 1–14, 2011.

[2] E. Galanti, Y. Kaspi, F. J. Simons, D. Durante, M. Parisi, and S. J. Bolton, "Determining the depth of Jupiter's great red spot with Juno: A Slepian approach," *The Astrophysical Journal Letters*, vol. 874, no. 2, Apr. 2019.

[3] F. A. Dahlen and F. J. Simons, "Spectral estimation on a sphere in geophysics and cosmology," *Geophys. J. Int.*, vol. 174, pp. 774–807, Sep. 2008.

[4] D. Marinucci, D. Pietrobon, A. Balbi, P. Baldi, P. Cabella, G. Kerkyacharian, P. Natoli, D. Picard, and N. Vittorio, "Spherical needlets for cosmic microwave background data analysis," *Mon. Not. R. Astron. Soc.*, vol. 383, no. 2, pp. 539–545, 2008.

[5] F. J. Simons and F. Dahlen, "Spherical Slepian functions and the polar gap in geodesy," *Geophys. J. Int.*, vol. 166, no. 3, pp. 1039–1061, 2006.

[6] P.-M. Lam, C.-S. Leung, and T.-T. Wong, "Noise-resistant fitting for spherical harmonics," *IEEE Transactions on Visualization and Computer Graphics*, vol. 12, pp. 254–265, 2006.

[7] S. Nadeem, Z. Su, W. Zeng, A. Kaufman, and X. Gu, "Spherical parameterization balancing angle and area distortions," *IEEE Transactions on Visualization and Computer Graphics*, vol. 23, no. 6, pp. 1663–1676, 2016.

[8] M. K. Chung, K. M. Dalton, L. Shen, A. C. Evans, and R. J. Davidson, "Weighted Fourier series representation and its application to quantifying the amount of gray matter," *IEEE Trans. Med. Imag.*, vol. 26, no. 4, pp. 566–581, Apr. 2007.

[9] A. P. Bates, Z. Khalid, and R. A. Kennedy, "An optimal dimensionality sampling scheme on the sphere with accurate and efficient spherical harmonic transform for diffusion MRI," *IEEE Signal Process. Lett.*, vol. 23, no. 1, pp. 15–19, Jan. 2016.

[10] W. Zhang, M. Zhang, R. A. Kennedy, and T. D. Abhayapala, "On high-resolution head-related transfer function measurements: An efficient sampling scheme," *IEEE Trans. Audio, Speech, Language Process.*, vol. 20, no. 2, pp. 575–584, 2012.

[11] A. P. Bates, Z. Khalid, and R. A. Kennedy, "Novel sampling scheme on the sphere for head-related transfer function measurements," *IEEE/ACM Trans. Audio, Speech, Language Process.*, vol. 23, no. 6, pp. 1068–1081, Jun. 2015.

[12] Y. F. Alem, Z. Khalid, and R. A. Kennedy, "3D spatial fading correlation for uniform angle of arrival distribution," *IEEE Commun. Lett.*, vol. 19, no. 6, pp. 1073–1076, Jun. 2015.

[13] F. Bashar, T. D. Abhayapala, and S. A. Salehin, "Dimensionality of spatio-temporal broadband signals observed over finite spatial and temporal windows," *IEEE Transactions on Wireless Communications*, vol. 15, no. 10, pp. 6758–6770, Oct. 2016.

[14] F. Hlawatsch, G. Matz, H. Kirchauer, and W. Kozek, "Time-frequency formulation, design, and implementation of time-varying optimal filters for signal estimation," *IEEE Trans. Signal Process.*, vol. 48, no. 5, pp. 1417–1432, 2000.

[15] I. Sasgen, Z. Martinec, and K. Fleming, "Wiener optimal filtering of GRACE data," *Stud. Geophys. Geod.*, vol. 50, no. 4, pp. 499–508, 2006.

[16] B. T. T. Yeo, W. Ou, and P. Golland, "On the construction of invertible filter banks on the 2-sphere," *IEEE Trans. Image Process.*, vol. 17, no. 3, pp. 283–300, Mar. 2008.

[17] J. D. McEwen, M. P. Hobson, and A. N. Lasenby, "Optimal filters on the sphere," *IEEE Trans. Signal Process.*, vol. 56, no. 8, pp. 3813–3823, Aug. 2008.

[18] R. Klees, E. Revtova, B. Gunter, P. Ditmar, E. Oudman, H. Winsemius, and H. Savenije, "The design of an optimal filter for monthly GRACE gravity models," *Geophys. J. Int.*, vol. 175, no. 2, pp. 417–432, 2008.

[19] R. Arora and H. Parthasarathy, "Optimal estimation and detection in homogeneous spaces," *IEEE Trans. Signal Process.*, vol. 58, no. 5, pp. 2623–2635, 2010.

[20] J. D. McEwen, S. M. Feeney, M. C. Johnson, and H. V. Peiris, "Optimal filters for detecting cosmic bubble collisions," *Phys. Rev. D.*, vol. 85, no. 10, pp. 103–502, 2012.

[21] Z. Khalid, P. Sadeghi, R. A. Kennedy, and S. Durrani, "Spatially varying spectral filtering of signals on the unit sphere," *IEEE Trans. Signal Process.*, vol. 61, no. 3, pp. 530–544, Feb. 2013.

[22] P. Sadeghi, R. A. Kennedy, and Z. Khalid, "Minimum mean square error equalization on the 2-sphere," in *2014 IEEE Workshop on Statistical Signal Processing (SSP)*. IEEE, 2014, pp. 101–104.

[23] J.-L. Starck and A. Bijaoui, "Filtering and deconvolution by the wavelet transform," *Signal Processing*, vol. 35, pp. 195–211, 1994.

[24] O. Renaud, J.-L. Starck, and F. Murtagh, "Wavelet-based combined signal filtering and prediction," *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, vol. 35, no. 6, pp. 1241–1251, Dec. 2005.

[25] S. G. Mallat, "A theory for multiresolution signal decomposition: the wavelet representation," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 11, no. 7, pp. 674–693, Jul. 1989.

[26] I. Daubechies, "The wavelet transform, time-frequency localization and signal analysis," *IEEE Trans. Inf. Theory*, vol. 36, no. 5, pp. 961–1005, Sep. 1990.

[27] S. G. Mallat, *A Wavelet Tour of Signal Processing*, 3rd ed. Massachusetts, USA: Academic Press, 2009.

[28] F. J. Narcowich and J. D. Ward, "Non-stationary wavelets on the m-sphere for scattered data," *Appl. Comput. Harm. Anal.*, vol. 3, pp. 324–336, 1996.

[29] F. Freeden and U. Windheuser, "Combined spherical harmonic and wavelet expansion – a future concept in the earth's gravitational determination," *Appl. Comput. Harm. Anal.*, vol. 4, pp. 1–37, 1997.

[30] J.-P. Antoine and P. Vandergheynst, "Wavelets on the n-sphere and related manifolds," *J. Math. Phys.*, vol. 39, pp. 3987–4008, 1998.

[31] ——, "Wavelets on the 2-sphere: A group-theoretical approach," *Appl. Comput. Harm. Anal.*, vol. 7, no. 3, pp. 262–291, 1999.

[32] Y. Wiaux, L. Jacques, and P. Vandergheynst, "Correspondence principle between spherical and Euclidean wavelets," *Astrophys. J.*, vol. 632, no. 1, pp. 15–28, Oct. 2005.

[33] J.-L. Starck, Y. Moudden, P. Abrial, and M. Nguyen, "Wavelets, ridgelets and curvelets on the sphere," *Astron. & Astrophys.*, vol. 446, pp. 1191–1204, Feb. 2006.

[34] Y. Wiaux, J. D. McEwen, P. Vandergheynst, and O. Blanc, "Exact reconstruction with directional wavelets on the sphere," *Mon. Not. R. Astron. Soc.*, vol. 388, no. 2, pp. 770–788, 2008.

[35] J. D. McEwen, C. Durastanti, and Y. Wiaux, "Localisation of directional scale-discretised wavelets on the sphere," *Appl. Comput. Harm. Anal.*, vol. 44, no. 1, pp. 59–88, Jan 2018.

[36] J.-L. Starck and F. Murtagh, "Multiscale entropy filtering," *Signal Processing*, vol. 76, no. 2, pp. 147–165, Jul. 1999.

[37] B. Leistedt, J. D. McEwen, P. Vandergheynst, and Y. Wiaux, "S2let: A code to perform fast wavelet analysis on the sphere," *Astron. & Astrophys.*, vol. 558, no. A128, pp. 1–9, 2013.

[38] R. A. Kennedy and P. Sadeghi, *Hilbert Space Methods in Signal Processing*. Cambridge, UK: Cambridge University Press, Mar. 2013.

[39] P. J. Kostelec and D. N. Rockmore, "FFTs on the rotation group," *J. Fourier Anal. and Appl.*, vol. 14, pp. 145–179, 2008.

[40] N. J. Vilenkin, *Special Functions and the Theory of Group Representations, Volume 22 of Translations of mathematical monographs*. Rhode Island, USA: American Mathematical Soc., 1968.

[41] D. Marinucci and G. Peccati, *Random Fields on the Sphere: Representation, Limit Theorems and Cosmological Applications*. Cambridge University Press, 2011.

[42] Z. Khalid, R. A. Kennedy, P. Sadeghi, and S. Durrani, "Spatio-spectral formulation and design of spatially-varying filters for signal estimation on the 2-sphere," in *Proc. Wavelets and Sparsity XV, SPIE international symposium on optics and photonics, invited contribution*, San Diego, California, United States, Aug 2013.

[43] W. Freeden and M. Schreiner, "Non-orthogonal expansions on the sphere," *Math. Methods in the Appl. Sciences*, vol. 18, no. 2, pp. 83–120, Feb. 1995.