Implications for compressed sensing of a new sampling theorem on the sphere

Jason D. McEwen*, Gilles Puy*, Jean-Philippe Thiran*, Pierre Vandergheynst*, Dimitri Van De Ville*† and Yves Wiaux*†
* Ecole Polytechnique Fédérale de Lausanne (EPFL), CH-1015 Lausanne, Switzerland
† University of Geneva (UniGE), CH-1211 Geneva, Switzerland

Sampling theorems on the sphere state that all the information of a continuous band-limited signal on the sphere may be contained in a discrete set of samples. For an equiangular sampling of the sphere, the Driscoll & Healy (DH) [1] sampling theorem has become the standard, requiring $\sim 4L^2$ samples on the sphere to represent exactly a signal band-limited in its spherical harmonic decomposition at L. Recently, a new sampling theorem on an equiangular grid has been developed by McEwen & Wiaux (MW) [2], requiring only $\sim 2L^2$ samples to represent exactly a band-limited signal, thereby redefining Nyquist rate sampling on the sphere. No sampling theorem on the sphere reaches the optimal number of samples suggested by the L^2 dimension of a band-limited signal in harmonic space (although the MW sampling theorem comes closest to this bound). A reduction by a factor of two in the number of samples required to represent a band-limited signal on the sphere between the DH and MW sampling theorems has important implications for compressed sensing.

Compressed sensing on the sphere has been studied recently for signals sparse in harmonic space [3], where a discrete grid on the sphere is not required. However, for signals sparse in the spatial domain (or in its gradient) a discrete grid on the sphere is essential. A reduction in the number of samples of the grid required to represent a band-limited signal improves both the dimensionality and sparsity of the signal, which in turn affects the quality of reconstruction.

We illustrate the impact of the number of samples of the DH and MW sampling theorems with an inpainting problem, where measurements are made in the spatial domain (as dictated by many applications). A test signal sparse in its gradient is constructed from a binary Earth map, smoothed to give a signal band-limited at $L = 32$. We first solve the total variation (TV) inpainting problem directly on the sphere:

$$\boldsymbol{x}^\star = \underset{\boldsymbol{x}}{\arg\min} \, \|\boldsymbol{x}\|_{\rm TV} \;\text{ such that }\; \|\boldsymbol{y} - \Phi\boldsymbol{x}\|_2 \leq \epsilon \,, \qquad (1)$$

where M noisy measurements $\boldsymbol{y}$ of the signal $\boldsymbol{x}$ are made. The measurement operator Φ represents a random masking of the signal. The TV norm $\|\cdot\|_{\rm TV}$ is defined to approximate the continuous TV norm on the sphere and thus includes the quadrature weights of the adopted sampling theorem, regularising the gradient computed on the sphere. However, as discussed, the dimensionality of the signal $\boldsymbol{x}$ is optimal in harmonic space. Consequently, we reduce the dimensionality of our problem by recovering the harmonic coefficients $\hat{\boldsymbol{x}}$ directly:

$$\hat{\boldsymbol{x}}^\star = \underset{\hat{\boldsymbol{x}}}{\arg\min} \, \|\Psi\hat{\boldsymbol{x}}\|_{\rm TV} \;\text{ such that }\; \|\boldsymbol{y} - \Phi\Psi\hat{\boldsymbol{x}}\|_2 \leq \epsilon \,, \qquad (2)$$

where Ψ represents the inverse spherical harmonic transform; the signal on the sphere is recovered by $\boldsymbol{x}^\star = \Psi\hat{\boldsymbol{x}}^\star$. For this problem the dimensionality of the signal directly recovered $\hat{\boldsymbol{x}}$ is identical for both sampling theorems, however sparsity in the spatial domain remains superior (*i.e.* fewer non-zero values) for the MW sampling theorem.

This work is supported by CIBM of the Geneva and Lausanne Universities, EPFL, and the SNSF, Leenaards and Louis-Jeantet foundations.

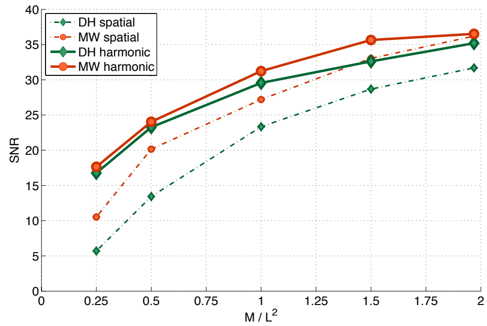
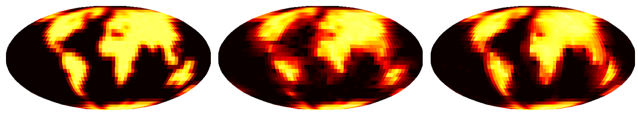


Fig. 1. Reconstruction performance for the DH and MW sampling theorems



(a) Ground truth (b) DH reconstruction (c) MW reconstruction

Fig. 2. Reconstructed Earth topographic data for $M/L^2 = 1/2$

Reconstruction performance is plotted in Fig. 1 when solving the inpainting problem in the spatial (1) and harmonic (2) domains, for both sampling theorems (averaged over ten simulations of random measurement operators and independent and identically distributed Gaussian noise). Strictly speaking, compressed sensing corresponds to the range $M/L^2 < 1$ when considering the harmonic representation of the signal. Nevertheless, we extend our tests to $M/L^2 \sim 2$, corresponding to the equivalent of Nyquist rate sampling on the MW grid. In all cases the superior performance of the MW sampling theorem is evident. In Fig. 2 we show example reconstructions, where the superior quality of the MW reconstruction is again clear.

Although recovering the signal in the harmonic domain is more effective, it is also computationally more demanding. At present we are thus limited to low band-limits. To solve the convex optimisation problem in the harmonic domain both the inverse spherical harmonic transform and its adjoint operator are required. A fast inverse spherical harmonic transform exists [2], from which a fast adjoint operator follows directly. The application of fast inverse and adjoint operators is the focus of ongoing research and will allow compressed sensing problems on the sphere to be tackled effectively at much higher band-limits.

References

[1] J. R. Driscoll and D. M. J. Healy, "Computing Fourier transforms and convolutions on the sphere," *Advances in Applied Mathematics*, vol. 15, pp. 202–250, 1994.

[2] J. D. McEwen and Y. Wiaux, "A novel sampling theorem on the sphere," *IEEE Trans. Sig. Proc.*, in press, 2011.

[3] H. Rauhut and R. Ward, "Sparse recovery for spherical harmonic expansions," *ArXiv:1102.4097*, 2011.